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Classification of lepton mixing matrices from finite residual symmetries

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Abstract: Assuming that neutrinos are Majorana particles, we perform a complete classification of all possible mixing matrices which are fully determined by residual symmetries in the charged-lepton and neutrino mass matrices. The classification is based on the assumption that the residual symmetries originate from a *finite* flavour symmetry group. The mathematical tools which allow us to accomplish this classification are theorems on sums of roots of unity. We find 17 sporadic cases plus one infinite series of mixing matrices associated with three-flavour mixing, all of which have already been discussed in the literature. Only the infinite series contains mixing matrices which are compatible with the data at the 3 sigma level.

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1 Introduction

Residual symmetries in the mass matrices are a means to determine rows and columns in the mixing matrix U as pure numbers, independent of the values of the masses, by assuming finiteness of the underlying flavour symmetry group G —for recent reviews on group theory see for instance [1–5]. This model-independent approach contrasts, for instance, with the idea of texture zeros, where mass ratios are related to mixing angles and CP phases. In recent years, assuming neutrinos are Majorana particles, much effort has gone into the discussion of the lepton mixing matrices U from residual symmetries. Though a lot of results have been achieved [6–24], the present status has not been reached in a systematic way.[1] Thus it is by no means clear if the cases existing in the literature encompass all relevant mixing matrices. In the present paper we close this gap by accomplishing a complete classification of all possible U completely determined by residual symmetries. It turns out that indeed all relevant mixing matrices can already be found in the literature.

Before we delve into the advertised classification, we shortly review the idea of residual symmetries, which also allows us to introduce our notation. The mass Lagrangian — obtained through flavour symmetry breaking — has the form

$$\mathcal{L}_{\text{mass}} = -\bar{\ell}_L M_\ell \ell_R + \frac{1}{2}\nu_L^T C^{-1} \mathcal{M}_\nu \nu_L + \text{H.c.}, \tag{1.1}$$

where C is the charge-conjugation matrix, ℓ_L and ℓ_R contain the three left and right-handed charged-lepton fields, respectively, and ν_L consists of the three left-handed neutrino fields. The neutrino mass matrix $\mathcal{M}_\nu$ is symmetric, due to the assumed Majorana nature of the neutrinos, but complex in general. Residual symmetries in the mass matrices are defined via

$$T^\dagger M_\ell M_\ell^\dagger T = M_\ell M_\ell^\dagger, \quad S^T \mathcal{M}_\nu S = \mathcal{M}_\nu, \tag{1.2}$$

where T and S are unitary matrices. In the Standard Model, the fields ν_L and ℓ_L are together in a weak doublet, and we assume that this is also true for any extension of it. Therefore, T and S are given in the same weak basis. The idea of residual symmetries is that any matrix T and any matrix S of equation (1.2) contributes to the complete flavour symmetry group G of some unknown extension of the Standard Model. The group G is broken in the charged-lepton sector to the group G_ℓ with $T \in G_\ell$ and in the neutrino sector

[1] Some work has also been done in relation to the quark sector [25–27] and on SO(10) GUTs [28].

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to G_ν with $S \in G_\nu$. Because of the specific form of the mass term in the case of Majorana neutrinos, G_ν must be a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ [6–18], or

$$G_\nu \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2, \tag{1.3}$$

the Klein four-group, if we confine ourselves to matrices S with $\det S = 1$.

In this paper we want to discuss all possibilities such that the residual symmetry groups G_ℓ and G_ν fully determine the lepton mixing matrix U. Thus we identify G_ν with the Klein four-group:

$$G_\nu = \{\mathbb{1},\, S_1,\, S_2,\, S_3\} \tag{1.4}$$

with three commuting matrices S_j with the properties $S_j^2 = \mathbb{1}$ and $S_j S_k = S_l$ for $j \neq k \neq l \neq j$. As we will see in the following, in most cases G_ℓ is generated by a single matrix T. In some cases, however, one needs two matrices T_1, T_2 in order to fully determine U; in such instances, one of these matrices alone fixes only a row in U.

How can the residual symmetries determine the mixing matrix U? Let us denote the diagonalising matrices of $M_\ell M_\ell^\dagger$ and $\mathcal{M}_\nu$ by U_ℓ and U_ν, respectively. Then

$$U_\ell^\dagger M_\ell M_\ell^\dagger U_\ell = \mathrm{diag}\left(m_e^2, m_\mu^2, m_\tau^2\right), \quad U_\nu^T \mathcal{M}_\nu U_\nu = \mathrm{diag}\left(m_1, m_2, m_3\right) \tag{1.5}$$

and the lepton mixing matrix is given by

$$U = U_\ell^\dagger U_\nu. \tag{1.6}$$

Since

$$S_j^T \mathcal{M}_\nu S_j = \mathcal{M}_\nu \quad \forall\, j = 1, 2, 3, \tag{1.7}$$

one can show that U_ν not only diagonalises $\mathcal{M}_\nu$, but also the matrices S_j. Therefore, the requirement that $U_\nu^\dagger S_j U_\nu$ is diagonal for $j = 1, 2, 3$ already determines U_ν and no knowledge of $\mathcal{M}_\nu$ is necessary in this framework. The same argument applies to $M_\ell M_\ell^\dagger$ and T. The diagonalisation of T determines U_ℓ, if T has non-degenerate eigenvalues. The only other non-trivial case, differing from the previous one, is that of two commuting T_1, T_2, which after diagonalisation have the form

$$\hat{T}_1 = \mathrm{diag}\,(\lambda_1', \lambda_1, \lambda_1), \quad \hat{T}_2 = \mathrm{diag}\,(\lambda_2, \lambda_2', \lambda_2) \tag{1.8}$$

or permutations thereof. In a nutshell, in the framework of residual symmetries, U_ν and U_ℓ are determined by G_ν and G_ℓ, respectively, whence one gets a handle on the lepton mixing matrix U.[2]

However, the framework set forth above is still too general to be treatable. An important extra ingredient is that the group G generated by G_ν and G_ℓ is *finite*. This is an *ad hoc* assumption, but it has the important consequence that the eigenvalues of all $T \in G_\ell$, the eigenvalues of all products of the generators of G, e.g. $S_j T$, and the eigenvalues of all multiple products of the generators are roots of unity.[3] By taking traces of these matrices

[2] If U is only partially determined by the residual symmetries, then either a column or a row is fixed [24].

[3] The eigenvalues of the S_j are ± 1, thus they are trivially roots of unity.

we obtain sums of roots of unity, and therefore known mathematical results concerning this type of sums are applicable — for a previous application see [29], which form the basis for our classification of all cases of U. Note that in order for the group G to be finite it is necessary but in general not sufficient that the eigenvalues of its generators are roots of unity. Nevertheless, in the cases discussed in this paper, it turns out that, if the group generators and some of their products have finite order, then G is finite.

In summary, the classification of possible mixing matrices U is based on the following premises:

i. The Standard Model with three families of leptons is the low-energy model of some theory of lepton flavour.

ii. Neutrinos have Majorana nature.

iii. The residual group G_ν is the Klein four-group, i.e. $G_\nu = \mathbb{Z}_2 \times \mathbb{Z}_2$.

iv. The group G, generated by the elements of G_ℓ and G_ν, is finite.

v. The mixing matrix U is completely determined by the residual symmetries.

In the following, we will always work in a basis where the S_j are diagonal:

$$S_1 = \mathrm{diag}\,(1, -1, -1), \quad S_2 = \mathrm{diag}\,(-1, 1, -1), \quad S_3 = \mathrm{diag}\,(-1, -1, 1). \tag{1.9}$$

In this convenient basis, U_ν is an unknown diagonal matrix of phase factors. Therefore, in the framework of residual symmetries, the Majorana phases in U are indeterminate and the lepton mixing matrix U is simply given by

$$U = U_\ell^\dagger \tag{1.10}$$

up to rephasings. Moreover, since in our framework the order of the charged-lepton and neutrino masses is undefined, it is only possible to determine U up to independent row and column permutations.

The paper is organized as follows. In section 2 we discuss some mathematical results. These are used later, in section 3, to determine the possible forms of $|T|$ which is defined as the matrix of absolute values of T:

$$|T|_{ij} = |T_{ij}| \quad \forall\, i, j = 1, 2, 3. \tag{1.11}$$

It turns out that there are only five such basic forms (later on reduced to three by additional considerations), modulo independent permutations of rows and columns. Then, in section 4, we perform a general discussion of equivalent forms of T, which are those forms which lead to trivial variations of the mixing matrix U. In particular, we investigate the freedom of permutations and argue that, without loss of generality, we can confine ourselves to matrices T which can be written as

$$T = \widetilde{T}\hat{\kappa}. \tag{1.12}$$

In this formula, $\widetilde{T}$ contains the "internal" or CKM-type phase of a unitary matrix and $\hat{\kappa}$ is a diagonal matrix of phase factors. We identify in section 5 the genuinely different forms of $|T|$ which emerge from the five basic forms by permutations. It remains to investigate the phases of T. We first determine in section 6 the "internal" phase of T for each form of $|T|$. In section 7 we demonstrate that forms 1 and 4 do not lead to finite groups. The extensive section 8 is devoted to the computation of $\hat{\kappa}$ or "external" phases in T for the basics forms 2, 3 and 5. At this point we have completely determined T up to equivalent forms, so in this section we also provide our main result, the possible cases of U. In section 9 we discuss the remaining solutions where G_ℓ has two generators, each with a twofold degenerate eigenvalue — see equation (1.8). We conclude with section 10. Lengthy calculational details are deferred to appendices.

2 Mathematical tools

Vanishing sums of roots of unity. We will make use of three theorems related to roots of unity. The first theorem concerns vanishing sums of roots of unity. Some remarks are appropriate before we reproduce the theorem of Conway and Jones (theorem 6 in [30]). Formal sums of roots of unity with rational coefficients form a ring [30]. A sum of roots of unity $\mathcal{S}'$ is *similar* to a sum of roots of unity $\mathcal{S}$ if there is a rational number q and a root of unity δ such that $\mathcal{S}' = q\delta\mathcal{S}$. The length of a sum of roots of unity $\mathcal{S}$ is the number of distinct roots involved. Note, however, that $-\alpha = (-1)\alpha$ for any root of unity α, i.e. $-\alpha$ does *not* count separately for the length of $\mathcal{S}$; in other words, $\alpha + (-\alpha)$ has length zero. The roots occurring in the following theorem are

$$\omega = e^{2\pi i/3}, \quad \beta = e^{2\pi i/5}, \quad \gamma = e^{2\pi i/7}. \tag{2.1}$$

Note that

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \beta = \frac{\sqrt{5}-1}{4} + i\sqrt{\frac{5+\sqrt{5}}{8}}, \tag{2.2}$$

while γ expressed in radicals is too complicated to be shown here.

Theorem 1 (Conway and Jones) *Let $\mathcal{S}$ be a non-empty vanishing sum of length at most 9. Then either $\mathcal{S}$ involves θ, $\theta\omega$ and $\theta\omega^2$ for some root θ, or $\mathcal{S}$ is similar to one of*

a) $1 + \beta + \beta^2 + \beta^3 + \beta^4$,

b) $-\omega - \omega^2 + \beta + \beta^2 + \beta^3 + \beta^4$,

c) $1 + \gamma + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 + \gamma^6$,

d) $1 + \beta + \beta^4 - (\omega + \omega^2)(\beta^2 + \beta^3)$,

e) $-\omega - \omega^2 + \gamma + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 + \gamma^6$,

f) $\beta + \beta^4 - (\omega + \omega^2)(1 + \beta^2 + \beta^3)$,

g) $1 + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 - (\omega + \omega^2)(\gamma + \gamma^6)$,

h) $1 - (\omega + \omega^2)(\beta + \beta^2 + \beta^3 + \beta^4)$.

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Non-vanishing sums of roots of unity. The second theorem concerns sums of roots of unity with values on the unit circle in the complex plane.

Theorem 2 *Let ζ be an n-th root of unity, i.e. $\zeta = e^{2\pi i/n}$, and let $\mathcal{S} = \sum_{k=0}^{n-1} a_k \zeta^k$ be a sum with integer coefficients a_k. If $|\mathcal{S}| = 1$, then $\mathcal{S}$ is itself a root of unity.*

A proof of this theorem can for instance be deduced from lemma 1.6 in [31] and a discussion of this issue can be found on [32].

The link between theorem 2 and the problems in the present paper is provided by the following theorem.

Theorem 3 *Let G be a finite group with $T \in G_\ell$ and let c be one of the numbers $1/2$, $(\sqrt{5}+1)/4$ or $(\sqrt{5}-1)/4$. Moreover, T_{jj} is a diagonal element and T_{kl}, T_{lk} are off-diagonal elements of T. Then the following holds:*

$$|T_{jj}| = c \;\Rightarrow\; T_{jj} = c\xi, \quad |T_{kl}T_{lk}| = \frac{1}{4} \;\Rightarrow\; T_{kl}T_{lk} = \frac{\xi'}{4} \tag{2.3}$$

with roots of unity ξ, ξ'.

Proofs. It is easy to show that

$$\operatorname{Tr}(TS_j) + \operatorname{Tr} T = 2T_{jj}. \tag{2.4}$$

Since T, $TS_j \in G$ and G is finite, the eigenvalues of T and TS_j must be roots of unity. Therefore, the traces of these matrices are sums over three roots of unity and $2T_{jj}$ is a sum over six roots of unity. Thus, if $2|T_{jj}| = 1$, theorem 2 tells us that $2T_{jj}$ is a root of unity. Now suppose that $2|T_{jj}| = (\sqrt{5}+1)/2$ or $(\sqrt{5}-1)/2$. Since

$$\left(\frac{\sqrt{5}+1}{2}\right)^{-1} = \beta + \beta^4 \quad \text{and} \quad \left(\frac{\sqrt{5}-1}{2}\right)^{-1} = -\beta^2 - \beta^3\,, \tag{2.5}$$

we find that both

$$2T_{jj}\left(\frac{\sqrt{5}+1}{2}\right)^{-1} \quad \text{and} \quad 2T_{jj}\left(\frac{\sqrt{5}-1}{2}\right)^{-1} \tag{2.6}$$

are normalized sums over roots of unity. Again, theorem 2 applies. This finishes the proof of the first part of the theorem. For the second part we note that, since T is a unitary matrix,

$$\left(T^{-1}\right)_{jj} = \frac{1}{\det T}\left(T_{kk}T_{ll} - T_{kl}T_{lk}\right) = (T_{jj})^* \tag{2.7}$$

for $j \neq k \neq l \neq j$. We know that $\det T$ is a product of three roots of unity and that $2T_{kk}$, $2T_{ll}$ are sums over roots of unity. Therefore, $4T_{kl}T_{lk}$ is a sum over roots of unity as well. Now we apply once more theorem 2 and the proof is finished. □

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Root of unity or not, that is the question. The next theorem addresses the problem of finding out whether a complex number ζ with $|\zeta| = 1$ is a root of unity or not. We can answer this question if we know a polynomial $P(x)$ with rational coefficients such that $P(\zeta) = 0$. Because then from $P(x)$ we can determine the minimal polynomial $m_\zeta(x)$ of ζ, which is defined as being irreducible over the rational numbers and normalized.[4] Since the minimal polynomial is unique the following statement holds.

Theorem 4 *Let ζ be a complex number with $|\zeta| = 1$ and $m_\zeta(x)$ its minimal polynomial. Then ζ is a root of unity if and only if $m_\zeta(x)$ is a cyclotomic polynomial.*

Note that cyclotomic polynomials are the minimal polynomials of roots of unity. There is a straightforward corollary to theorem 4 which makes use of the fact that cyclotomic polynomials have integer coefficients.

Theorem 5 *If there are non-integer coefficients in $m_\zeta(x)$, then ζ is not a root of unity.*

3 The basic forms of $|T|$

In this section we determine the basic forms of $|T|$ which follow from our assumptions listed in the introduction.

We begin by considering the matrices

$$Y^{(ij)} = T^\dagger S_i T S_j. \tag{3.1}$$

We argue that these matrices have eigenvalues 1, $\lambda^{(ij)}$, $(\lambda^{(ij)})^*$.
Proof. The matrices $Y^{(ij)}$ fulfill

$$\det Y^{(ij)} = 1, \quad S_j^{-1} Y^{(ij)} S_j = \left(Y^{(ij)}\right)^\dagger. \tag{3.2}$$

Therefore, the complex conjugate of every eigenvalue of $Y^{(ij)}$ is also an eigenvalue. Moreover, the product of the three eigenvalues must be one. So the spectrum of $Y^{(ij)}$ contains one complex eigenvalue $\lambda^{(ij)}$, its complex conjugate and 1. $\square$
Since

$$\sum_{j=1}^{3} S_j = -\mathbb{1}, \tag{3.3}$$

it follows that

$$\sum_{k=1}^{3} \operatorname{Tr} Y^{(kj)} = 1 \quad \text{and} \quad \sum_{k=1}^{3} \operatorname{Tr} Y^{(ik)} = 1. \tag{3.4}$$

Written in terms of the eigenvalues, these relations are

$$\sum_{k=1}^{3} \left(\lambda^{(kj)} + \lambda^{(kj)*}\right) + 2 = 0 \quad \text{and} \quad \sum_{k=1}^{3} \left(\lambda^{(ik)} + \lambda^{(ik)*}\right) + 2 = 0 \tag{3.5}$$

for $i, j = 1, 2, 3$. These equations can be tackled with theorem 1.

[4]The means that the coefficient of its highest power is 1.

Suppose we have obtained the possible $\lambda^{(ij)}$. Defining

$$t_{kl} \equiv |T_{kl}|^2\,, \tag{3.6}$$

we observe that the unitarity of T gives

$$\sum_{k,l} t_{kl} = 3. \tag{3.7}$$

Furthermore, we obtain

$$\begin{aligned}
\operatorname{Tr} Y^{(ij)} &= 1 + \lambda^{(ij)} + (\lambda^{(ij)})^* \\
&= \sum_{k,l} t_{kl}\,(S_i)_{kk}\,(S_j)_{ll} \\
&= \sum_{k\neq i}\sum_{l\neq j} t_{kl} + t_{ij} - \sum_{k\neq i} t_{kj} - \sum_{l\neq j} t_{il} = \\
&= 3 - 2\sum_{k\neq i} t_{kj} - 2\sum_{l\neq j} t_{il} \\
&= -1 + 4t_{ij}.
\end{aligned} \tag{3.8}$$

In this way we arrive at the relation

$$|T_{ij}|^2 = \frac{1}{2}\left(1 + \operatorname{Re}\lambda^{(ij)}\right). \tag{3.9}$$

Thus in order to obtain $|T_{ij}|$ we are left with the task of finding all possible solutions of the generic equation

$$\sum_{k=1}^{3} (\lambda_k + \lambda_k^*) + 2 = 0, \tag{3.10}$$

with roots of unity λ_1, λ_2, λ_3. These solutions are derived in appendix A through application of theorem 1. It turns out that, up to permutations and complex conjugations, equation (3.10) has only three solutions:

$$(\lambda_1,\,\lambda_2,\,\lambda_3) = \left\{\begin{array}{cl} (i,\,\omega,\,\omega) & \text{(A)}, \\ (\omega,\,\beta,\,\beta^2) & \text{(B)}, \\ (-1,\,\lambda,\,-\lambda) & \text{(C)}, \end{array}\right. \tag{3.11}$$

where $\lambda = e^{i\vartheta}$ is an arbitrary root of unity.

The details of the tedious manipulations for finding all possible forms of the matrix t are given in appendix B. Here we mention only that it is important to take into account that t stems from a unitary matrix — see inequality (B.6), which rules out many cases. The surprising result of appendix B is that, up to permutations of rows and columns, there are only five basic forms of t. In the following, however, we will rather use $|T|$ instead of t. Therefore, we display here the basic forms of $|T|$ obtained by inserting the solutions (3.11) into equation (3.9):

$$\text{Form 1:}\quad |T| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{3.12a}$$

$$\text{Form 2:}\quad |T| = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \tag{3.12b}$$

$$\text{Form 3:}\quad |T| = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix}, \tag{3.12c}$$

$$\text{Form 4:}\quad |T| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \end{pmatrix}, \tag{3.12d}$$

$$\text{Form 5:}\quad |T| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{3.12e}$$

Note that form 5 derives from equation (B.16) via $\sin^2\theta = (1 - \cos\vartheta)/2$ and, therefore, $\theta = \pm\vartheta/2 + k\pi$ with an arbitrary integer k. Since we know that ϑ is a rational angle, i.e. a rational multiple of π, it follows that θ must be a rational angle too.

4 Equivalent forms of T

It is expedient to reflect on the ambiguities in the determination of U with the method used here.

The first observation is that the choice of basis for the S_j in equation (1.9) leaves still some freedom for basis transformations. The reason is that

$$S_j \to V^\dagger S_j V \quad \text{with} \quad V = P\hat{\sigma}, \tag{4.1}$$

where P is a 3×3 permutation matrix and $\hat{\sigma}$ is a diagonal matrix of phase factors, may transform one S_j into another, but the set of matrices of equation (1.9) is invariant under this similarity transformation. On the other hand, the transformation (4.1) acts also on T as

$$T \to V^\dagger T V, \tag{4.2}$$

and we can use this freedom to fix some conventions for T. This is an important issue because it prevents us from over-counting the number of cases and from studying equivalent cases twice. Two matrices T, T', which are connected via $T' = V^\dagger T V$, are called equivalent in the following.

The freedom in reordering and rephasing expressed by equation (4.2) can be used for a strategy to determine all inequivalent forms of T.

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1. For each of the five basic forms of equation (3.12), we take the matrix displayed there as the starting point and indicate it by the subscript A.

2. Due to equation (4.2) we are allowed to confine ourselves to permutations from the right in order to find inequivalent matrices $|T|_I$ with $I = B, C, \ldots$

3. In general there will be less than six inequivalent matrices $|T|$ for each of the basics forms, as some matrices which emerge from each other by a permutation from the right might still be equivalent due to the equality of some matrix elements or, in the case of form 5, because it is possible to make the exchange $\cos\theta \leftrightarrow \sin\theta$.

4. After having found, for each basic form, the inequivalent matrices $|T|_I = |T|_A P_I$, where P_I is a permutation matrix, we determine the internal phase associated with each $|T|_I$. Denoting the resulting matrix by $\widetilde{T}_I$, we can choose phase conventions such that $\widetilde{T}_I = \widetilde{T}_A P_I$ for all I.

5. The matrix T will also have external phases. Applying again equation (4.2), we can assume that these phases are taken care of by multiplying $\widetilde{T}$ by a diagonal matrix of phase factors $\hat{\kappa}$ from the right. In essence, the $\hat{\kappa}$ will be determined by the requirement that T has finite order. Note, however, that, given a basic form and one of its possible $\widetilde{T}_I$, there can be several solutions of $\hat{\kappa}$.

6. Finally, since the matrices S_j are real, for every solution T there is the complex conjugate solution T^*.

Eventually we are not interested in the possible T but in the possible mixing matrices U. According to equation (1.10) the two matrices are linked via

$$T = U^\dagger \hat{T} U \quad \text{with} \quad \hat{T} = \operatorname{diag}\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right). \tag{4.3}$$

The order of the eigenvalues is indeterminate and $\hat{T}$ is invariant under phase transformations. Therefore, U can undergo rephasing and permutations from the left. Equation (4.2) implies that the same holds from the right, with rephasings and permutations independent from those on the left. To this indeterminacy one has to add the possibility of complex conjugation of U. Given these trivial phase variations of the mixing matrix and the need to factor them out, we will be focusing on $|U|^2$.

The group G, determined by the three S_j and by one or two matrices T which generate G_ℓ, is not changed by the above manipulations of T. Actually, what we have at hand is not directly the group G but its representation on the three leptonic gauge doublets. In this sense, the change $T \to T^*$ for the $T \in G_\ell$ corresponds to switching from one representation to its complex conjugate. We can also remove an overall phase factor ξ from a T, if ξ is root of unity. This will, in general, change the group, but the group will remain finite.

5 Permutations of the basic forms

In order to investigate the inequivalent permutations of the columns of $|T|_A$, it is appropriate to use the representation of the permutations $p \in S_3$ as permutation matrices:

$$p \to M(p) = \left(e_{p(1)}, e_{p(2)}, e_{p(3)}\right) \quad \text{with} \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.1}$$

It is easy to check that the representation property $M(pp') = M(p)M(p')$ is fulfilled for any two p, $p' \in S_3$. The basic relation we need is given by

$$M(q)^T |T|_A M(q) = |T|_A \quad \Rightarrow \quad M(q)^T \Big(|T|_A M(p)\Big) M(q) = |T|_A \Big(M(q)^T M(p) M(q)\Big). \tag{5.2}$$

Therefore, invariance of $|T|_A$ under some $q \in S_3$ means that $|T|_A M(p_1)$ and $|T|_A M(p_2)$ are equivalent provided that permutations p_1, p_2 are related through conjugation by q, i.e. $p_2 = q^{-1} p_1 q$. Now we discuss all five basic forms under this aspect.

It is easy to see that $|T|_A$ of form 1, equation (3.12a), is invariant under *all* $q \in S_3$. Since S_3 has three conjugacy classes, there are three inequivalent forms. Apart from $|T|_A$, we choose $|T|_B \equiv |T|_A M\,[(23)]$ and $|T|_C \equiv |T|_A M\,[(132)]$.

Turning to form 2, equation (3.12b), $|T|_A$ is obviously invariant under $q = (23)$. Given that $|T|_A\, M\,[(12)] = |T|_A\, M\,[(132)]$, $q^{-1}(12)q = (13)$, and $q^{-1}(132)q = (123)$, we can conclude that the only other inequivalent form is $|T|_B \equiv |T|_A M\,[(12)]$.

Concerning form 3, equation (3.12c), we find that it is invariant under cyclic permutations. Since one transposition is transformed into the other two by conjugation with the two cyclic permutations, we can for instance choose $|T|_B \equiv |T|_A M\,[(23)]$, as representative of the transpositions. Finally, cyclic permutations give the inequivalent forms $|T|_C \equiv |T|_A M\,[(132)]$ and $|T|_D \equiv |T|_A M\,[(123)]$.

The matrix $|T|_A$ of form 4, equation (3.12d), is invariant under $q = (23)$. In contrast to form 2, all columns are different. Given that $q^{-1}(12)q = (13)$ and $q^{-1}(123)q = (132)$, we are lead to a choice of inequivalent forms $|T|_B \equiv |T|_A M\,[(23)]$, $|T|_C \equiv |T|_A M\,[(12)]$ and $|T|_D \equiv |T|_A M\,[(132)]$.

Since in form 5, equation (3.12e), the angle θ is free apart from being rational, the exchange $\cos\theta \leftrightarrow \sin\theta$ relates equivalent forms. So in effect we are lead back to the argumentation applied to form 2. For definiteness, we choose $|T|_B \equiv |T|_A M\,[(123)]$.

In summary, the inequivalent forms obtained by the above discussion are given by

$$\text{Form 1:} \quad |T|_A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad |T|_B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}, \tag{5.3a}$$

$$|T|_C = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

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$$\text{Form 2:}\quad |T|_A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad |T|_B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}, \tag{5.3b}$$

$$\text{Form 3:}\quad |T|_A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix}, \quad |T|_B = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} & \frac{1}{2} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & \frac{\sqrt{5}+1}{4} \end{pmatrix}, \tag{5.3c}$$

$$|T|_C = \begin{pmatrix} \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \end{pmatrix}, \quad |T|_D = \begin{pmatrix} \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \end{pmatrix},$$

$$\text{Form 4:}\quad |T|_A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \end{pmatrix}, \qquad |T|_B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \end{pmatrix}, \tag{5.3d}$$

$$|T|_C = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \end{pmatrix}, \qquad |T|_D = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & \frac{\sqrt{5}+1}{4} \end{pmatrix},$$

$$\text{Form 5:}\quad |T|_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad |T|_B = \begin{pmatrix} 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \end{pmatrix}. \tag{5.3e}$$

6 The internal phase of T

In order to compute the internal phase of T, we can use the formulas provided in [33]:

$$R = \mathrm{Re}\,(T_{11}T_{22}T_{12}^*T_{21}^*) = \frac{1}{2}\,(1 - t_{11} - t_{22} - t_{12} - t_{21} + t_{11}t_{22} + t_{12}t_{21})\,, \tag{6.1}$$

$$J = \mathrm{Im}\,(T_{11}T_{22}T_{12}^*T_{21}^*) = \left(t_{11}t_{22}t_{12}t_{21} - R^2\right)^{1/2}. \tag{6.2}$$

It is well known that the existence of an internal phase of a unitary matrix is independent of the phase convention [34]. This can also be seen from the above equations for R and J. However, where to place the internal phase in T is, of course, convention-dependent. It suffices to find the internal phase for subform A in equation (5.3) for each of the five basic forms, since the other subforms emerge from A by permutation of the columns.

For each form we choose a suitable set of elements T_{jk} of T which are real and positive by convention. This set contains three of the elements occurring in R and J. Then we apply equations (6.1) and (6.2) and compute the phase of the remaining element. From this, taking into account that the columns of a unitary matrix form an orthonormal system, the matrix $\widetilde{T}$, defined in equation (1.12), is obtained. Since these procedures are standard methods in linear algebra, we only display the results:

$$\text{Form 1:}\quad \widetilde{T}_A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2}\varphi \\ \frac{1}{2}\varphi^2 & \frac{1}{\sqrt{2}} & -\frac{1}{2}\varphi^2 \\ \frac{1}{2}\varphi & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{6.3a}$$

$$\text{Form 2:}\quad \widetilde{T}_A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \tag{6.3b}$$

$$\text{Form 3:}\quad \widetilde{T}_A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix}, \tag{6.3c}$$

$$\text{Form 4:}\quad \widetilde{T}_A = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho_0 & \frac{1}{2}\omega & \frac{1}{2}\omega^2 \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{1}{2}\omega^2 & \frac{\sqrt{5}+1}{4}\omega & \frac{\sqrt{5}-1}{4} \end{pmatrix}, \tag{6.3d}$$

$$\text{Form 5:}\quad \widetilde{T}_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{6.3e}$$

Of the phase factors occurring in these formulas, ω is defined in equation (2.1) and

$$\varphi = \frac{1-i\sqrt{7}}{\sqrt{8}}, \quad \rho_0 = \frac{\sqrt{5}+i\sqrt{3}}{\sqrt{8}}. \tag{6.4}$$

As discussed in section 4, $\widetilde{T}$ only needs to be computed modulo complex conjugation.

7 Forms 1 and 4 do not lead to finite groups

7.1 Form 1

We begin with

$$\text{Form 1A:}\quad T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2}\varphi \\ \frac{1}{2}\varphi^2 & \frac{1}{\sqrt{2}} & -\frac{1}{2}\varphi^2 \\ \frac{1}{2}\varphi & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \operatorname{diag}(\kappa_1, \kappa_2, \kappa_3). \tag{7.1}$$

Theorem 3 requires that the phase factors of $T_{12}T_{21}$, $T_{13}T_{31}$ and $T_{23}T_{32}$ are roots of unity. Therefore, these phases are $\kappa_1\kappa_2\varphi^2 \equiv \xi'_{12}$, $\kappa_1\kappa_3\varphi^2 \equiv \xi'_{13}$ and $\kappa_2\kappa_3\varphi^2 \equiv \xi'_{23}$, respectively, with roots of unity ξ'_{jk} $(j<k)$. From these equations we derive $\kappa_1^2 = \varphi^{-2}\xi'_{12}\xi'_{13}/\xi'_{23}$, etc. Hence it follows that $\kappa_j = \varphi^{-1}\xi_j$ with roots of unity ξ_j. Furthermore,

$$(\det T)^2 = x\,(\xi_1\xi_2\xi_3)^2 \quad \text{with} \quad x = \frac{1+3i\sqrt{7}}{8}. \tag{7.2}$$

Since T has finite order, x has to be a root of unity. But one can easily check that x fulfills

$$x^2 - \frac{1}{4}x + 1 = 0. \tag{7.3}$$

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Therefore, according to theorem 5, x is not a root of unity and T of equation (7.1) does not belong to a finite group.

Next we consider

$$\text{Form 1B:} \quad T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2}\varphi & \frac{1}{2} \\ \frac{1}{2}\varphi^2 & -\frac{1}{2}\varphi^2 & \frac{1}{\sqrt{2}} \\ \frac{1}{2}\varphi & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \operatorname{diag}(\kappa_1, \kappa_2, \kappa_3)\,. \tag{7.4}$$

Applying again theorem 3, we find that the phase factors of $T_{12}T_{21}$, T_{22} and T_{33} are roots of unity. From this it is easy to show that $\kappa_1 = \varphi^{-1}\xi_1$, $\kappa_2 = \varphi^{-2}\xi_2$ and $\kappa_3 = \xi_3$ with roots of unity ξ_j. Now we proceed as with form 1A and obtain again equation (7.2), which excludes form 1B as well.

Finally we discuss

$$\text{Form 1C:} \quad T = \begin{pmatrix} \frac{1}{2}\varphi & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2}\varphi^2 & \frac{1}{2}\varphi^2 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2}\varphi & -\frac{1}{2} \end{pmatrix} \operatorname{diag}(\kappa_1, \kappa_2, \kappa_3)\,. \tag{7.5}$$

Here theorem 3 tells us that the phase factors of T_{11}, T_{22} and T_{33} are roots of unity and one obtains the same relations for κ_j as for form 1B. Computing the determinant of T, we are again lead to equation (7.2). Hence form 1C is excluded.

7.2 Form 4

We proceed analogously to form 1. We first consider

$$\text{Form 4A:} \quad T = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho_0 & \frac{1}{2}\omega & \frac{1}{2}\omega^2 \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{1}{2}\omega^2 & \frac{\sqrt{5}+1}{4}\omega & \frac{\sqrt{5}-1}{4} \end{pmatrix} \operatorname{diag}(\kappa_1, \kappa_2, \kappa_3)\,. \tag{7.6}$$

Theorem 3 requires that the phase factors of $T_{12}T_{21}$, T_{22} and T_{33} are roots of unity, therefore, the κ_j must all be roots of unity. Computing the determinant of T, we obtain

$$\det T = \chi\kappa_1\kappa_2\kappa_3 \quad \text{with} \quad \chi = \frac{1}{8}\left[1 + 3\sqrt{5} - i\sqrt{3}\left(\sqrt{5} - 1\right)\right]. \tag{7.7}$$

Since T has finite order, its determinant is a root of unity and so is χ. Then also

$$\chi\omega = \frac{1}{4}\left(-1 + i\sqrt{15}\right) \tag{7.8}$$

must be root of unity. But $\chi\omega$ is a root of the equation

$$x^2 + \frac{1}{2}x + 1 = 0. \tag{7.9}$$

Therefore, according to theorem 5, it cannot be a root of unity, which is a contradiction. Hence form 4A does not lead to a finite group.

Now we consider

$$\text{Form 4B:}\quad T = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho_0 & \frac{1}{2}\omega^2 & \frac{1}{2}\omega \\ \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \\ \frac{1}{2}\omega^2 & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4}\omega \end{pmatrix} \operatorname{diag}(\kappa_1, \kappa_2, \kappa_3)\,. \tag{7.10}$$

In the same way as for form 1A, we find that all κ_j are roots of unity. In this case

$$\det T = -\chi\kappa_1\kappa_2\kappa_3 \tag{7.11}$$

and the argument excluding this form goes through as before.

Next we discuss

$$\text{Form 4C:}\quad T = \begin{pmatrix} \frac{1}{2}\omega & \frac{1}{\sqrt{2}}\rho_0 & \frac{1}{2}\omega^2 \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4}\omega & \frac{1}{2}\omega^2 & \frac{\sqrt{5}-1}{4} \end{pmatrix} \operatorname{diag}(\kappa_1, \kappa_2, \kappa_3)\,. \tag{7.12}$$

The determinant of this form is equal to the one of form 4B, and the phases κ_j have to be roots of unity as before. So one can exclude this form as well.

The final case is

$$\text{Form 4D:}\quad T = \begin{pmatrix} \frac{1}{2}\omega^2 & \frac{1}{\sqrt{2}}\rho_0 & \frac{1}{2}\omega \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2}\omega^2 & \frac{\sqrt{5}+1}{4}\omega \end{pmatrix} \operatorname{diag}(\kappa_1, \kappa_2, \kappa_3)\,. \tag{7.13}$$

Again, all κ_j must be roots of unity and the determinant is the same as for form 4A.

It is interesting to note that forms 1 and 4, the only two of the basic five forms which have a non-trivial internal phase, do not lead to finite groups. Thus we do not discuss them further.

8 The external phases of T and the resulting mixing matrices

It remains to investigate the remaining forms 2, 3 and 5. We still have to determine the external phases $\hat{\kappa} = \operatorname{diag}(\kappa_1, \kappa_2, \kappa_3)$ of $T = \widetilde{T}\hat{\kappa}$ for a total of eight subforms of $\widetilde{T}$:

$$\text{Form 2:}\quad \widetilde{T}_A,\quad \widetilde{T}_B = \widetilde{T}_A\, M[(12)], \tag{8.1a}$$

$$\text{Form 3:}\quad \widetilde{T}_A,\quad \widetilde{T}_B = \widetilde{T}_A\, M[(23)],\quad \widetilde{T}_C = \widetilde{T}_A\, M[(132)],\quad \widetilde{T}_D = \widetilde{T}_A\, M[(123)], \tag{8.1b}$$

$$\text{Form 5:}\quad \widetilde{T}_A,\quad \widetilde{T}_B = \widetilde{T}_A\, M[(123)], \tag{8.1c}$$

which are read off from equations (5.3b) and (6.3b), equations (5.3c) and (6.3c), and equations (5.3e) and (6.3e), for forms 2, 3 and 5, respectively. All these forms have trivial internal phases.

The basic idea to determine the possible $\hat{\kappa}$ for each subform is the observation that, for any element $T' \in G_\ell$, the matrix $|T'|$ must be of one of the basics forms 2, 3 or 5, after having excluded forms 1 and 4. In particular, this holds for $T' = T^2$. Therefore, for any of the eight subforms under consideration the elements

$$\left|\left(T^2\right)_{jk}\right| = \left|\sum_{l=1}^{3} \widetilde{T}_{jl}\kappa_l \widetilde{T}_{lk}\right| \tag{8.2}$$

must belong to a matrix $P_1 |T|_A P_2$ where P_1 and P_2 are arbitrary permutation matrices. Two remarks are in order. Let us assume that we discuss for instance T of form 2.

i. Then T^2 can belong to form 2, 3 or 5.

ii. Since we have used up already the freedom of permutations for the subform we begin with, we have to admit *all* possible permutations P_1 and P_2.

In the following we will present, for each subform in an own subsection, the results for the phases $\hat{\kappa}$, the eigenvalues $\lambda_j^{(0)}$ ($j = 1, 2, 3$) of T and the matrix $|U|^2$. Every triple consisting of T, including its external phases, the eigenvalues of T and $|U|^2$ is called solution. The latter matrices, whose entries are the $|U_{jk}|^2$, represent the main result of this paper. The details of the computations for each subform are deferred to appendix C. Every solution obtains a tag $\mathcal{C}_i$ in the case of a *complete* determination of $|U|^2$ by a single T. In the case that T is degenerate and determines only one row of $|U|^2$, then the tag is $\mathcal{P}_i$ which stands for *partial*.

As we will see, in general, there are several solutions of external phases $\hat{\kappa}$ for each of the eight subforms of equation (8.1). Sometimes it occurs that for one $\widetilde{T}$ there are two matrices $\hat{\kappa}_a$ and $\hat{\kappa}_b$ of external phase such that the corresponding matrices T_a and T_b have the same $|U|^2$ and eigenvalues which are related by

$$\left(\lambda_{a1}^{(0)}, \lambda_{a2}^{(0)}, \lambda_{a3}^{(0)}\right) \propto \left(\lambda_{b1}^{(0)}, \lambda_{b2}^{(0)}, \lambda_{b3}^{(0)}\right) \quad \text{or} \quad \left(\lambda_{b1}^{(0)}, \lambda_{b2}^{(0)}, \lambda_{b3}^{(0)}\right)^* . \tag{8.3}$$

In this case, we assign only one solution tag.

8.1 Form 2A

In appendix C.1 it is shown that form 2A

$$T = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \operatorname{diag}\left(\kappa_1, \kappa_2, \kappa_3\right) \tag{8.4}$$

requires $\kappa_2 = \pm\kappa_3$ and thus leads to two subcases.

In the first subcase with solution tag $\mathcal{C}_1$, we have

$$\kappa_2 = \kappa_3, \quad \left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \left(\sqrt{\kappa_1\kappa_2}, -\sqrt{\kappa_1\kappa_2}, -\kappa_2\right) \tag{8.5}$$

and the mixing matrix

$$\mathcal{C}_1: \quad |U|^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{8.6}$$

The second subcase is given by

$$\kappa_2 = -\kappa_3, \quad \left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa\left(1, \omega, \omega^2\right), \tag{8.7}$$

$$\mathcal{C}_2: \quad |U|^2 = \frac{1}{3}\begin{pmatrix} 1 & 1 + \mathrm{Re}\,\sigma & 1 - \mathrm{Re}\,\sigma \\ 1 & 1 + \mathrm{Re}\,(\omega\sigma) & 1 - \mathrm{Re}\,(\omega\sigma) \\ 1 & 1 + \mathrm{Re}\left(\omega^2\sigma\right) & 1 - \mathrm{Re}\left(\omega^2\sigma\right) \end{pmatrix}. \tag{8.8}$$

For $\mathcal{C}_2$, the quantities κ and σ are roots of unity related to κ_1 and κ_2 by

$$\kappa^3 = -\kappa_1\kappa_2^2 \quad \text{and} \quad \sigma = -\kappa\kappa_2^*, \tag{8.9}$$

respectively. Note that the transformations $\sigma \to (-\omega)^x \sigma$ and $\sigma \to (-\omega)^x \sigma^*$ with $x = 0, \ldots, 5$ lead to a permutation of the mixing pattern in $|U|^2$, which accounts for 12 of the 36 possible row and column permutations. Once this permutation freedom is taken into account, it becomes clear that two roots of unity σ, σ' will yield the same mixing angles if and only if $\mathrm{Re}\left(\sigma^6\right) = \mathrm{Re}\left(\sigma'^6\right)$.

While $\mathcal{C}_1$ is known as bimaximal mixing, $\mathcal{C}_2$ corresponds to trimaximal mixing; for the specific choice of σ such that $\sigma^6 = 1$, tribimaximal mixing is obtained. It will turn out that $\mathcal{C}_2$ is the only series of mixing matrices genuinely involving the three flavours. From residual symmetries, $\mathcal{C}_2$ has for instance been derived in [20], while it was obtained in [21] from GAP [35] and the Small Groups Library [36]. Recently, this series has been accommodated in a model [37]. We note that the $|U|^2$ of $\mathcal{C}_2$ has a trivial CKM-type phase.

If $\kappa_1 = \kappa_2$ in $\mathcal{C}_1$, then $\lambda_2^{(0)} = \lambda_3^{(0)}$ and the residual symmetries fix only one row in the mixing matrix U:

$$\kappa_1 = \kappa_2 = \kappa_3, \quad \left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa_1\left(1, -1, -1\right), \tag{8.10}$$

with the mixing matrix

$$\mathcal{P}_1: \quad |U|^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{8.11}$$

The symbol "$\times$" indicates the positions in $|U|^2$ which are not fully determined by the T of equation (8.4) with $\hat{\kappa} \propto \mathbb{1}$.

8.2 Form 2B

According to the analysis carried out in appendix C.2, the external phases allowed for form 2B are

$$\hat{\kappa} = \mathrm{diag}\,(x, \pm x^*, 1)\,\kappa_3 \quad \text{with} \quad x = \pm\varphi,\ \pm\rho_0, \tag{8.12}$$

where φ and ρ_0 are defined in equation (6.4) and κ_3 is an arbitrary root of unity. Complex conjugation of $\hat{\kappa}$ leads to further allowed cases. However, as discussed earlier, this is a trivial variation of the solutions and we ignore it. Thus there are eight subcases of form 2B.

First subcase.

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \operatorname{diag}\left(-\varphi, -\varphi^*, 1\right) \kappa_3 \tag{8.13}$$

with eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa_3(-1, i, -i). \tag{8.14}$$

We denote this solution by $\mathcal{C}_3$. It gives the mixing matrix

$$\mathcal{C}_3: \quad |U|^2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8}\left(3+\sqrt{7}\right) & \frac{1}{8}\left(3-\sqrt{7}\right) & \frac{1}{4} \\ \frac{1}{8}\left(3-\sqrt{7}\right) & \frac{1}{8}\left(3+\sqrt{7}\right) & \frac{1}{4} \end{pmatrix}. \tag{8.15}$$

Second subcase.

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \operatorname{diag}\left(-\varphi, \varphi^*, 1\right) \kappa_3 \tag{8.16}$$

with eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa_3(\gamma^2, \gamma^4, \gamma). \tag{8.17}$$

In this case with the solution tag $\mathcal{C}_4$, the seventh root of unity, $\gamma = e^{2\pi i/7}$, occurs in the eigenvalues and the mixing matrix reads

$$\mathcal{C}_4: \quad |U|^2 = \begin{pmatrix} r_1 & r_2 & r_3 \\ r_2 & r_3 & r_1 \\ r_3 & r_1 & r_2 \end{pmatrix}, \tag{8.18}$$

where the r_i are the roots of the equation

$$-1 + 14x - 56x^2 + 56x^3 = 0. \tag{8.19}$$

Their approximate numerical values are $r_1 = 0.664$, $r_2 = 0.204$, $r_3 = 0.132$.

Third subcase.

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \operatorname{diag}\left(\varphi, \varphi^*, 1\right) \kappa_3 \tag{8.20}$$

with eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = -\kappa_3(1, \omega^2, \omega) \tag{8.21}$$

leads to solution $\mathcal{C}_5$ with the mixing matrix

$$\mathcal{C}_5: \quad |U|^2 = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{12}\left(5-\sqrt{21}\right) & \frac{1}{12}\left(5+\sqrt{21}\right) & \frac{1}{6} \\ \frac{1}{12}\left(5+\sqrt{21}\right) & \frac{1}{12}\left(5-\sqrt{21}\right) & \frac{1}{6} \end{pmatrix}. \tag{8.22}$$

Fourth subcase.

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \mathrm{diag}\,(\varphi, -\varphi^*, 1)\,\kappa_3 \tag{8.23}$$

with eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa_3(\gamma^5, \gamma^3, \gamma^6) \tag{8.24}$$

does not produce a new mixing matrix, but repeats the mixing matrix of solution $\mathcal{C}_4$ of the second subcase. Moreover, the eigenvalues of T of the second and fourth subcases are related by complex conjugation. Therefore, according to the philosophy put forward in the beginning of section 8, we do not assign a new solution tag here.

Fifth subcase.

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \mathrm{diag}\,(\rho_0, \rho_0^*, 1)\,\kappa_3 \tag{8.25}$$

with eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = -\kappa_3(\beta^3, \beta^2, 1). \tag{8.26}$$

In this case, the fifth root of unity, $\beta = e^{2\pi i/5}$, appears. The mixing matrix of this solution is given by

$$\mathcal{C}_6 : \quad |U|^2 = \begin{pmatrix} \frac{1}{8}\left(3 + \frac{1}{\sqrt{5}} + \sqrt{6 + \frac{6}{\sqrt{5}}}\right) & \frac{1}{8}\left(3 + \frac{1}{\sqrt{5}} - \sqrt{6 + \frac{6}{\sqrt{5}}}\right) & \frac{1}{20}\left(5 - \sqrt{5}\right) \\ \frac{1}{8}\left(3 + \frac{1}{\sqrt{5}} - \sqrt{6 + \frac{6}{\sqrt{5}}}\right) & \frac{1}{8}\left(3 + \frac{1}{\sqrt{5}} + \sqrt{6 + \frac{6}{\sqrt{5}}}\right) & \frac{1}{20}\left(5 - \sqrt{5}\right) \\ \frac{1}{20}\left(5 - \sqrt{5}\right) & \frac{1}{20}\left(5 - \sqrt{5}\right) & \frac{1}{10}\left(5 + \sqrt{5}\right) \end{pmatrix}. \tag{8.27}$$

Sixth subcase.

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \mathrm{diag}\,(\rho_0, -\rho_0^*, 1)\,\kappa_3 \tag{8.28}$$

with eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa_3\omega\,(i, 1, -i). \tag{8.29}$$

Here we obtain the mixing matrix

$$\mathcal{C}_7 : \quad |U|^2 = \begin{pmatrix} \frac{1}{16}\left(5 - \sqrt{3} + \sqrt{5} - \sqrt{15}\right) & \frac{1}{16}\left(5 - \sqrt{3} - \sqrt{5} + \sqrt{15}\right) & \frac{1}{8}\left(3 + \sqrt{3}\right) \\ \frac{1}{8}\left(3 - \sqrt{5}\right) & \frac{1}{8}\left(3 + \sqrt{5}\right) & \frac{1}{4} \\ \frac{1}{16}\left(5 + \sqrt{3} + \sqrt{5} + \sqrt{15}\right) & \frac{1}{16}\left(5 + \sqrt{3} - \sqrt{5} - \sqrt{15}\right) & \frac{1}{8}\left(3 - \sqrt{3}\right) \end{pmatrix}. \tag{8.30}$$

Seventh subcase.

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \mathrm{diag}\left(-\rho_0, -\rho_0^*, 1\right) \kappa_3 \tag{8.31}$$

with eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = -\kappa_3 (\beta^4, \beta, 1). \tag{8.32}$$

Here again the fifth root of unity, β, occurs. The corresponding mixing matrix is

$$\mathcal{C}_8 : \quad |U|^2 = \begin{pmatrix} \frac{1}{8}\left(3 - \frac{1}{\sqrt{5}} - \sqrt{6 - \frac{6}{\sqrt{5}}}\right) & \frac{1}{8}\left(3 - \frac{1}{\sqrt{5}} + \sqrt{6 - \frac{6}{\sqrt{5}}}\right) & \frac{1}{20}\left(5 + \sqrt{5}\right) \\ \frac{1}{8}\left(3 - \frac{1}{\sqrt{5}} + \sqrt{6 - \frac{6}{\sqrt{5}}}\right) & \frac{1}{8}\left(3 - \frac{1}{\sqrt{5}} - \sqrt{6 - \frac{6}{\sqrt{5}}}\right) & \frac{1}{20}\left(5 + \sqrt{5}\right) \\ \frac{1}{20}\left(5 + \sqrt{5}\right) & \frac{1}{20}\left(5 + \sqrt{5}\right) & \frac{1}{10}\left(5 - \sqrt{5}\right) \end{pmatrix}. \tag{8.33}$$

Eighth subcase.

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix} \mathrm{diag}\left(-\rho_0, \rho_0^*, 1\right) \kappa_3 \tag{8.34}$$

with eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa_3\, \omega^2 \,(-i, 1, i). \tag{8.35}$$

These eigenvalues are proportional to the complex conjugate ones of solution $\mathcal{C}_7$, sixth subcase, and the mixing matrix is also the same. Thus we do not assign a new solution tag.

8.3 Form 3A

According to the analysis in appendix C.3, the possible external phases for form 3A are

$$\hat{\kappa} = \mathrm{diag}\left(1, \omega, \omega^2\right) \kappa_1,\ \mathrm{diag}\left(-\omega^2, \omega, 1\right) \kappa_3,\ \mathrm{diag}\left(\omega, -\omega^2, 1\right) \kappa_3,\ \mathrm{diag}\left(\omega, 1, -\omega^2\right) \kappa_2 \tag{8.36}$$

with arbitrary roots of unity κ_1, κ_2 and κ_3.

First subcase.

$$T = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix} \mathrm{diag}\left(1, \omega, \omega^2\right) \kappa_1 \tag{8.37}$$

with eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa_1\, (1, \omega, \omega^2). \tag{8.38}$$

The corresponding mixing matrix is

$$\mathcal{C}_9 : \quad |U|^2 = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}. \tag{8.39}$$

Second subcase. Next we consider

$$T = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix} \operatorname{diag}\left(-\omega^2, \omega, 1\right) \kappa_3 \tag{8.40}$$

with eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa_3\, \omega^2\, (-i, -1, i). \tag{8.41}$$

This subcase provides the mixing matrix

$$\mathcal{C}_{10}: \quad |U|^2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{8}\left(3+\sqrt{3}\right) & \frac{1}{8}\left(3-\sqrt{3}\right) \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8}\left(3-\sqrt{3}\right) & \frac{1}{8}\left(3+\sqrt{3}\right) \end{pmatrix}. \tag{8.42}$$

According to equation (8.36) there are two more subcases to consider, however, it turns out that these again lead to the eigenvalues and the mixing matrix of $\mathcal{C}_{10}$.

8.4 Form 3B

According to the analysis in appendix C.4, there are four different cases of external phases:

$$\hat{\kappa} = \operatorname{diag}\,(1,1,-1)\,\kappa_1,\ \operatorname{diag}\,(1,1,1)\,\kappa_1,\ \operatorname{diag}\,(1,-1,1)\,\kappa_1,\ \operatorname{diag}\,(-1,1,1)\,\kappa_3, \tag{8.43}$$

with arbitrary roots of unity κ_1, κ_3.

First subcase.

$$T = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}+1}{4} & -\frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} & \frac{1}{2} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & -\frac{\sqrt{5}+1}{4} \end{pmatrix} \operatorname{diag}\,(1,1,-1)\,\kappa_1 \tag{8.44}$$

has the eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa_1\ \left(1, \beta, \beta^4\right) \tag{8.45}$$

and provides the mixing matrix

$$\mathcal{C}_{11}: \quad |U|^2 = \begin{pmatrix} \frac{1}{10}\left(5-\sqrt{5}\right) & 0 & \frac{1}{10}\left(5+\sqrt{5}\right) \\ \frac{1}{20}\left(5+\sqrt{5}\right) & \frac{1}{2} & \frac{1}{20}\left(5-\sqrt{5}\right) \\ \frac{1}{20}\left(5+\sqrt{5}\right) & \frac{1}{2} & \frac{1}{20}\left(5-\sqrt{5}\right) \end{pmatrix}. \tag{8.46}$$

Second subcase. The next subcase is

$$T = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}+1}{4} & -\frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} & \frac{1}{2} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & -\frac{\sqrt{5}+1}{4} \end{pmatrix} \operatorname{diag}\,(1,1,1)\,\kappa_1. \tag{8.47}$$

The eigenvalues are

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = -\kappa_1\ \left(\omega, \omega^2, 1\right) \tag{8.48}$$

and the mixing matrix is given by

$$\mathcal{C}_{12}: \quad |U|^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{12}\left(3+\sqrt{5}\right) & \frac{1}{12}\left(3-\sqrt{5}\right) \\ \frac{1}{2} & \frac{1}{12}\left(3+\sqrt{5}\right) & \frac{1}{12}\left(3-\sqrt{5}\right) \\ 0 & \frac{1}{6}\left(3-\sqrt{5}\right) & \frac{1}{6}\left(3+\sqrt{5}\right) \end{pmatrix}. \tag{8.49}$$

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Third subcase.

$$T = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}+1}{4} & -\frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} & \frac{1}{2} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & -\frac{\sqrt{5}+1}{4} \end{pmatrix} \mathrm{diag}\,(1,-1,1)\,\kappa_1 \tag{8.50}$$

with eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa_1\, \left(\beta^2, \beta^3, 1\right) \tag{8.51}$$

gives the mixing matrix

$$\mathcal{C}_{13}: \quad |U|^2 = \begin{pmatrix} \frac{1}{20}\left(5-\sqrt{5}\right) & \frac{1}{20}\left(5+\sqrt{5}\right) & \frac{1}{2} \\ \frac{1}{20}\left(5-\sqrt{5}\right) & \frac{1}{20}\left(5+\sqrt{5}\right) & \frac{1}{2} \\ \frac{1}{10}\left(5+\sqrt{5}\right) & \frac{1}{10}\left(5-\sqrt{5}\right) & 0 \end{pmatrix}. \tag{8.52}$$

Solution $\mathcal{C}_{13}$ has the same mixing matrix as $\mathcal{C}_{11}$, however, its eigenvalues are not related by way of equation (8.3). Therefore, we count it as a separate solution.

Fourth subcase. The last subcase

$$T = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}+1}{4} & -\frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} & \frac{1}{2} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & -\frac{\sqrt{5}+1}{4} \end{pmatrix} \mathrm{diag}\,(-1,1,1)\,\kappa_3 \tag{8.53}$$

has two degenerate eigenvalues:

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = \kappa_3\,(1,-1,-1)\,. \tag{8.54}$$

Therefore, the mixing matrix is only partially determined:

$$\mathcal{P}_2: \quad |U|^2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{8}\left(3+\sqrt{5}\right) & \frac{1}{8}\left(3-\sqrt{5}\right) \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{8.55}$$

8.5 Form 3C

According to the analysis in appendix C.5 there are four solutions for the external phases:

$$\hat{\kappa} = \mathrm{diag}\,\left(\omega, -\omega^2, 1\right)\kappa_3,\ \mathrm{diag}\,\left(-\omega^2, \omega, 1\right)\kappa_3,\ \mathrm{diag}\,\left(\omega, \omega^2, 1\right)\kappa_3,\ \mathrm{diag}\,\left(\omega, 1, -\omega^2\right)\kappa_2, \tag{8.56}$$

with arbitrary roots of unity κ_2, κ_3.

First subcase. We begin with

$$T = \begin{pmatrix} -\frac{\sqrt{5}+1}{4} & \frac{1}{2} & -\frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} \end{pmatrix} \mathrm{diag}\,\left(\omega, -\omega^2, 1\right)\kappa_3, \tag{8.57}$$

which has the eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = -\kappa_3\,\left(1, \omega^2, \omega\right) \tag{8.58}$$

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and leads to the mixing matrix

$$\mathcal{C}_{14}: \quad |U|^2 = \begin{pmatrix} \frac{1}{12}\left(3-\sqrt{5}\right) & \frac{1}{12}\left(3-\sqrt{5}\right) & \frac{1}{6}\left(3+\sqrt{5}\right) \\ \frac{1}{12}\left(3-\sqrt{5}\right) & \frac{1}{6}\left(3+\sqrt{5}\right) & \frac{1}{12}\left(3-\sqrt{5}\right) \\ \frac{1}{6}\left(3+\sqrt{5}\right) & \frac{1}{12}\left(3-\sqrt{5}\right) & \frac{1}{12}\left(3-\sqrt{5}\right) \end{pmatrix}. \tag{8.59}$$

Second subcase. Next we tackle

$$T = \begin{pmatrix} -\frac{\sqrt{5}+1}{4} & \frac{1}{2} & -\frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} \end{pmatrix} \operatorname{diag}\left(-\omega^2, \omega, 1\right) \kappa_3. \tag{8.60}$$

This matrix has the eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = -\kappa_3 \left(\beta^4, \beta, 1\right) \tag{8.61}$$

and provides the mixing matrix

$$\mathcal{C}_{15}: \quad |U|^2 = \begin{pmatrix} \frac{1}{8}\left(3+\frac{1}{\sqrt{5}}-\sqrt{6+\frac{6}{\sqrt{5}}}\right) & \frac{1}{8}\left(3+\frac{1}{\sqrt{5}}+\sqrt{6+\frac{6}{\sqrt{5}}}\right) & \frac{1}{20}\left(5-\sqrt{5}\right) \\ \frac{1}{8}\left(3+\frac{1}{\sqrt{5}}+\sqrt{6+\frac{6}{\sqrt{5}}}\right) & \frac{1}{8}\left(3+\frac{1}{\sqrt{5}}-\sqrt{6+\frac{6}{\sqrt{5}}}\right) & \frac{1}{20}\left(5-\sqrt{5}\right) \\ \frac{1}{20}\left(5-\sqrt{5}\right) & \frac{1}{20}\left(5-\sqrt{5}\right) & \frac{1}{10}\left(5+\sqrt{5}\right) \end{pmatrix}. \tag{8.62}$$

According to equation (8.56) there are two more subcases with eigenvalues

$$\kappa_3\, \omega^2 \left(\beta^4, \beta, 1\right) \quad \text{and} \quad -\kappa_2\, \omega \left(\beta^4, \beta, 1\right), \tag{8.63}$$

respectively. It turns out that in both subcases the mixing matrix is that of $\mathcal{C}_{15}$. Since by choice of κ_3 and κ_2, respectively, we can even match the eigenvalues of T in equation (8.61), we do not count these two subcases as separate solutions.

8.6 Form 3D

According to appendix C.6 there are the four solutions

$$\hat{\kappa} = \left(-\omega, \omega^2, 1\right) \kappa_3,\ \operatorname{diag}\left(\omega, -\omega^2, 1\right) \kappa_3,\ \operatorname{diag}\left(\omega, \omega^2, 1\right) \kappa_3,\ \operatorname{diag}\left(\omega, 1, -\omega^2\right) \kappa_2, \tag{8.64}$$

with arbitrary roots of unity κ_2, κ_3, for the external phases.

First subcase. We begin with

$$T = \begin{pmatrix} -\frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ -\frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \end{pmatrix} \operatorname{diag}\left(-\omega, \omega^2, 1\right) \kappa_3, \tag{8.65}$$

which has the eigenvalues

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = -\kappa_3 \left(\omega, 1, \omega^2\right) \tag{8.66}$$

and provides the mixing matrix

$$\mathcal{C}_{16}: \quad |U|^2 = \begin{pmatrix} \frac{1}{6}\left(3-\sqrt{5}\right) & \frac{1}{12}\left(3+\sqrt{5}\right) & \frac{1}{12}\left(3+\sqrt{5}\right) \\ \frac{1}{12}\left(3+\sqrt{5}\right) & \frac{1}{12}\left(3+\sqrt{5}\right) & \frac{1}{6}\left(3-\sqrt{5}\right) \\ \frac{1}{12}\left(3+\sqrt{5}\right) & \frac{1}{6}\left(3-\sqrt{5}\right) & \frac{1}{12}\left(3+\sqrt{5}\right) \end{pmatrix}. \tag{8.67}$$

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Second subcase. Next we consider

$$T = \begin{pmatrix} -\frac{\sqrt{5}-1}{4} & -\frac{\sqrt{5}+1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ -\frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \end{pmatrix} \operatorname{diag}\left(\omega, -\omega^2, 1\right) \kappa_3. \tag{8.68}$$

Its eigenvalues are given by

$$\left(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}\right) = -\kappa_3 \left(\beta^3, \beta^2, 1\right) \tag{8.69}$$

and the corresponding mixing matrix is

$$\mathcal{C}_{17}: \quad |U|^2 = \begin{pmatrix} \frac{1}{8}\left(3 - \frac{1}{\sqrt{5}} - \sqrt{6 - \frac{6}{\sqrt{5}}}\right) & \frac{1}{8}\left(3 - \frac{1}{\sqrt{5}} + \sqrt{6 - \frac{6}{\sqrt{5}}}\right) & \frac{1}{20}\left(5 + \sqrt{5}\right) \\ \frac{1}{8}\left(3 - \frac{1}{\sqrt{5}} + \sqrt{6 - \frac{6}{\sqrt{5}}}\right) & \frac{1}{8}\left(3 - \frac{1}{\sqrt{5}} - \sqrt{6 - \frac{6}{\sqrt{5}}}\right) & \frac{1}{20}\left(5 + \sqrt{5}\right) \\ \frac{1}{20}\left(5 + \sqrt{5}\right) & \frac{1}{20}\left(5 + \sqrt{5}\right) & \frac{1}{10}\left(5 - \sqrt{5}\right) \end{pmatrix}. \tag{8.70}$$

The third and fourth subcase have the eigenvalues

$$\kappa_3\, \omega \left(\beta^3, \beta^2, 1\right) \quad \text{and} \quad -\kappa_2 \left(\beta^3, \beta^2, 1\right), \tag{8.71}$$

respectively. Thus they are those of the second subcase multiplied by a phase factor. Since they also reproduce the mixing matrix of the second subcase, we do not consider them further.

8.7 Form 5A

If T is of form 5A, then obviously we have two-flavour mixing, which is completely off from realistic lepton mixing. There is a rather large number of such two-flavour mixing solutions, namely complete solutions $\mathcal{C}_{18}$–$\mathcal{C}_{29}$, and partial solutions $\mathcal{P}_3$–$\mathcal{P}_{14}$ where one row of $|U|^2$ is determined. Since these solutions are not relevant from the physics point of view, we defer them to appendix E, where we list them for completeness.

8.8 Form 5B

We are considering here the matrix

$$T = \begin{pmatrix} 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{pmatrix} \hat{\kappa}, \tag{8.72}$$

where $e^{i\theta}$ is a root of unity. Denoting the eigenvalues of T by λ_j and those of TS_1, with S_1 defined in equation (1.9), by λ'_j, we find, by taking the determinants and traces,

$$\lambda_1\lambda_2\lambda_3 = \lambda'_1\lambda'_2\lambda'_3, \quad \sin\theta\, \kappa_2 = \lambda_1 + \lambda_2 + \lambda_3, \quad -\sin\theta\, \kappa_2 = \lambda'_1 + \lambda'_2 + \lambda'_3. \tag{8.73}$$

Thus we obtain a vanishing sum of six roots of unity

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda'_1 + \lambda'_2 + \lambda'_3 = 0. \tag{8.74}$$

According to theorem 1, the vanishing sum over six roots of unity could be similar to sum b), but this case can be excluded by the first relation of equation (8.73). Then, theorem 1 allows for the following two solutions of this equality:

$$\delta_1(1-1)+\delta_2(1-1)+\delta_3(1-1)=0, \tag{8.75a}$$
$$\sigma_1(1+\omega+\omega^2)+\sigma_2(1+\omega+\omega^2)=0, \tag{8.75b}$$

with arbitrary roots of unity δ_k ($k=1,2,3$) and σ_l ($l=1,2$).

First subcase. First we discuss solution (8.75a). The assignment

$$(\lambda_1,\lambda_2,\lambda_3)=(\delta_1,\delta_2,\delta_3), \quad (\lambda'_1,\lambda'_2,\lambda'_3)=(-\delta_1,-\delta_2,-\delta_3) \tag{8.76}$$

leads to a contradiction with the first relation in equation (8.73). So without loss of generality we are left with

$$(\lambda_1,\lambda_2,\lambda_3)=(\delta_1,\delta_2,-\delta_2), \quad (\lambda'_1,\lambda'_2,\lambda'_3)=(-\delta_1,\delta_3,-\delta_3). \tag{8.77}$$

Since $\sum_j \lambda_j = -\sum_j \lambda'_j = \delta_1$, this first subcase yields $\sin^2\theta = 1$. Choosing without loss of generality $\sin\theta = 1$, we end up with the two-flavour mixing case

$$T=\begin{pmatrix} 0 & 0 & \kappa_3 \\ 0 & \kappa_2 & 0 \\ -\kappa_1 & 0 & 0 \end{pmatrix} \tag{8.78}$$

with a mixing angle of 45°. Note that κ_2 and $\kappa_1\kappa_3$ are roots of unity. Since two-flavour mixing is dealt with in subsection 8.7, we do not assign an extra solution tag to this.

Second subcase. Now we discuss solution (8.75b). If we make the assignment

$$(\lambda_1,\lambda_2,\lambda_3)=\sigma_1(1,\omega,\omega^2), \quad (\lambda'_1,\lambda'_2,\lambda'_3)=\sigma_2(1,\omega,\omega^2), \tag{8.79}$$

it immediately follows that $\sum_j \lambda_j = \sum_j \lambda'_j = 0$ and, therefore, $\sin\theta=0$. Taking without loss of generality $\cos\theta = 1$, we obtain

$$T=\begin{pmatrix} 0 & 0 & \kappa_3 \\ \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \end{pmatrix} \tag{8.80}$$

and the mixing matrix

$$\mathcal{C}_{30}: \quad |U|^2 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{8.81}$$

Note that here the product $\kappa_1\kappa_2\kappa_3$ must be a root of unity.

It remains to discuss possibilities of assignment other than that of equation (8.79). Without loss of generality, we can confine ourselves to

$$(\lambda_1,\lambda_2,\lambda_3)=\left(\sigma_1\omega^i,\sigma_1\omega^k,\sigma_2\omega^p\right), \quad (\lambda'_1,\lambda'_2,\lambda'_3)=\left(\sigma_2\omega^m,\sigma_2\omega^n,\sigma_1\omega^l\right), \tag{8.82}$$

with $i = 1$, $k = 2$, $l = 3$ and permutations thereof and $m = 1$, $n = 2$, $p = 3$ and permutations thereof. With this assignment, the first relation in equation (8.73) requires

$$\sigma_1^2 \sigma_2 \omega^{i+k+p} = \sigma_1 \sigma_2^2 \omega^{m+n+l}. \tag{8.83}$$

Because of $\omega^{i+k+l} = \omega^{m+n+p} = 1$, $\omega^{2l} = \omega^{-l}$ and $\omega^{2p} = \omega^{-p}$, we arrive at

$$\sigma_1 \omega^l = \sigma_2 \omega^p \quad \Rightarrow \quad \begin{cases} (\lambda_1, \lambda_2, \lambda_3) = \sigma_1 \left(\omega^i, \omega^k, \omega^l\right), \\ (\lambda_1', \lambda_2', \lambda_3') = \sigma_2 \left(\omega^m, \omega^n, \omega^p\right). \end{cases} \tag{8.84}$$

Therefore, also for the general assignment in equation (8.82) we obtain $\sum_j \lambda_j = \sum_j \lambda_j' = 0$ and we are lead back to solution $\mathcal{C}_{30}$.

9 Combining two $\mathcal{P}$-type solutions

Here we discuss the cases where two generators $T_1, T_2 \in G_\ell$ are necessary in order to fully determine the mixing matrix U. This happens if each T_j has a twofold degenerate eigenvalue — see equation (1.8):

$$T_1 = U^\dagger \hat{T}_1 U, \quad T_2 = U^\dagger \hat{T}_2 U \quad \text{with} \quad \hat{T}_1 = \text{diag}\left(\lambda_1', \lambda_1, \lambda_1\right), \quad \hat{T}_2 = \text{diag}\left(\lambda_2, \lambda_2', \lambda_2\right) \tag{9.1}$$

and roots of unity λ_1, λ_1', λ_2, λ_2' fulfilling

$$\lambda_1 \neq \lambda_1', \quad \lambda_2 \neq \lambda_2'. \tag{9.2}$$

Actually, we can only hope to discover a new $|U|^2$, not already covered by the previous section, if *none* of the elements of G_ℓ is non-degenerate. Let us examine this point further. The product $T_1 T_2$ is degenerate if and only if $\lambda_1 \lambda_2' = \lambda_1' \lambda_2$ whence if follows that

$$\hat{T}_2 = \frac{\lambda_2}{\lambda_1} \text{diag}\left(\lambda_1, \lambda_1', \lambda_1\right). \tag{9.3}$$

Next we consider

$$\left(\hat{T}_1\right)^2 \hat{T}_2 = \lambda_1^2 \lambda_2 \,\text{diag}\left(\left(\frac{\lambda_1'}{\lambda_1}\right)^2, \frac{\lambda_1'}{\lambda_1}, 1\right). \tag{9.4}$$

The only possibility to avoid non-degeneracy in this matrix is

$$\lambda_1' = -\lambda_1 \tag{9.5}$$

and we end up with

$$\hat{T}_1 = \lambda_1' S_1, \quad \hat{T}_2 = -\lambda_2 S_2, \tag{9.6}$$

with the diagonal sign matrices S_j defined in equation (1.9). Finally, without changing the finiteness of G we can remove the phase factors λ_1' and $-\lambda_2$. Thus, the only case we have to investigate is when G_ℓ is a Klein four-group.

According to the discussion in section 3, we know that for every pair of indices i, j the matrix $U^\dagger S_i U S_j$ has eigenvalues 1, $\sigma^{(ij)}$, $\left(\sigma^{(ij)}\right)^*$, where $\sigma^{(ij)}$ is a root of unity. So we have the same situation as in section 3, with T replaced by U. Therefore, instead of $|T|$, $|U|$ itself must be of one of the five basic forms and we have the solutions

$$\mathcal{CD}_1 : \ |U|^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}, \tag{9.7}$$

$$\mathcal{CD}_2 : \ |U|^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \tag{9.8}$$

$$\mathcal{CD}_3 : \ |U|^2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{8}(3-\sqrt{5}) & \frac{1}{8}(3+\sqrt{5}) \\ \frac{1}{8}(3+\sqrt{5}) & \frac{1}{4} & \frac{1}{8}(3-\sqrt{5}) \\ \frac{1}{8}(3-\sqrt{5}) & \frac{1}{8}(3+\sqrt{5}) & \frac{1}{4} \end{pmatrix}, \tag{9.9}$$

$$\mathcal{CD}_4 : \ |U|^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8}(3-\sqrt{5}) & \frac{1}{8}(3+\sqrt{5}) \\ \frac{1}{4} & \frac{1}{8}(3+\sqrt{5}) & \frac{1}{8}(3-\sqrt{5}) \end{pmatrix}, \tag{9.10}$$

$$\mathcal{CD}_5 : \ |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta & \sin^2\theta \\ 0 & \sin^2\theta & \cos^2\theta \end{pmatrix}. \tag{9.11}$$

In the last form, $e^{i\theta}$ must be a root of unity.

Scanning the genuine three-flavour mixing solutions of section 8 we find that the $|U|^2$ of $\mathcal{C}_1$ is of form 2 of equation (3.12). Naturally, the $|U|^2$ of all solutions with T of form 5A are of form 5 as well. But a $|U|^2$ of form 1, 3 or 4 of equation (3.12) does not occur in the previous section. Therefore, $\mathcal{CD}_1$, $\mathcal{CD}_3$ and $\mathcal{CD}_4$ provide new mixing matrices. We note also that, even though the previous argument is not enough to prove that the $\mathcal{CD}_i$ solutions lead to finite groups, it can be easily checked by explicit construction of the groups generated by S_1, S_2, T_1, and T_2 that they are finite (as are the ones associated with the $\mathcal{C}_i$ and $\mathcal{P}_i$ solutions).

10 Conclusions

In this paper, assuming that neutrinos are Majorana particles, we have presented a complete classification of all possible lepton mixing matrices U such that $|U|^2$, defined as the matrix with the elements $|U_{ij}|^2$, is completely determined by residual symmetries. In this model-independent framework, the entries of $|U|^2$ are obtained as pure numbers, determined by group-theoretical considerations. Evidently, the resulting matrices $|U|^2$ are independent of any parameter of a possible underlying theory and of the lepton masses. In our analysis we used the *ad hoc* assumption that the flavour group G is finite, which allowed us to use suitable theorems related to sums of roots of unity.

We have found 22 solutions associated with a genuine three-flavour mixing matrix U, i.e. where all three flavours are mixed: $\mathcal{C}_1 - \mathcal{C}_{17}$, $\mathcal{C}_{30}$ and $\mathcal{CD}_1 - \mathcal{CD}_4$. Since four pairs of solutions produce the same mixing matrix—$(\mathcal{C}_1, \mathcal{CD}_2)$, $(\mathcal{C}_6, \mathcal{C}_{15})$, $(\mathcal{C}_8, \mathcal{C}_{17})$ and $(\mathcal{C}_{11}, \mathcal{C}_{13})$ — there are only 17 sporadic $|U|^2$ patterns and one infinite series denoted by $\mathcal{C}_2$. The precise groups formed by the generators of the residual symmetries G_ν and G_ℓ will depend on one or more phases which are free parameters; for example, the T generator can always be multiplied by an arbitrary root of unity. Nevertheless, with the help of GAP [35], one can reach the conclusion that for the 17 sporadic $|U|^2$ patterns, the minimal groups[5] are

- S_4 for $\mathcal{C}_1/\mathcal{CD}_2$,
- $\mathrm{PSL}\,(2,7)$ for $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$, $\mathcal{CD}_1$,
- $\Sigma\,(360\times 3)$ for $\mathcal{C}_6/\mathcal{C}_{15}$, $\mathcal{C}_7$, $\mathcal{C}_8/\mathcal{C}_{17}$, $\mathcal{C}_9$, $\mathcal{C}_{10}$, $\mathcal{C}_{14}$, $\mathcal{C}_{16}$, $\mathcal{CD}_4$,
- A_5 for $\mathcal{C}_{11}/\mathcal{C}_{13}$, $\mathcal{C}_{12}$, $\mathcal{CD}_3$,
- A_4 for $\mathcal{C}_{30}$.

At this point a comparison of our results with those in the literature is in order. The list of mixing matrices in [20] induced by PSL(2, 7) agrees exactly with our list above and the same holds true for those induced by A_5; moreover, this reference contains, among some mixing matrices of the series $\mathcal{C}_2$, also the cases $\mathcal{C}_1/\mathcal{CD}_2$ and $\mathcal{C}_{30}$ with S_4 and A_4, respectively. Finally, the list of mixing matrices above associated with $\Sigma\,(360\times 3)$ agrees with those given in [22] in table 5 (with labels XIa/XIb) and table 6 (with labels beginning with K). So we conclude that all our sporadic cases have already been discussed in the literature.

For the phenomenologically interesting infinite series denoted by $\mathcal{C}_2$ where

$$|U|^2 = \frac{1}{3}\begin{pmatrix} 1 & 1+\mathrm{Re}\,\sigma & 1-\mathrm{Re}\,\sigma \\ 1 & 1+\mathrm{Re}\,(\omega\sigma) & 1-\mathrm{Re}\,(\omega\sigma) \\ 1 & 1+\mathrm{Re}\,(\omega^2\sigma) & 1-\mathrm{Re}\,(\omega^2\sigma) \end{pmatrix} \tag{10.1}$$

for some root of unity $\sigma = \exp(2i\pi p/n)$ with p coprime to n, we have discussed minimal groups and their generators in appendix D. Our result is

- $\Delta(6m^2)$ with $m = \mathrm{lcm}(6,n)/3$ when $9\nmid n$,
- $\left(\mathbb{Z}_m\times\mathbb{Z}_{m/3}\right)\rtimes S_3$ with $m=\mathrm{lcm}(2,n)$ when $9\mid n$.

Once all row and column permutations are considered, it turns out that in the four-dimensional space of the quadruples $\left\{\sin^2\theta_{12},\sin^2\theta_{23},\sin^2\theta_{13},\cos^2\delta\right\}$ the sporadic $|U|^2$ patterns yield 228 distinct values — see figure 1 for a graphical representation of the mixing angles.[6] All of them are excluded at 3 sigma by current neutrino oscillation data. If one leaves out $\cos^2\delta$, there are still 212 distinct points in the space of the triples $\left\{\sin^2\theta_{12},\sin^2\theta_{23},\sin^2\theta_{13}\right\}$.

[5]These are the smallest groups possible. In addition, for the cases $\mathcal{C}_{3-17}$, $\mathcal{CD}_1$, $\mathcal{CD}_3$ and $\mathcal{CD}_4$ it can be shown that the full flavour group must always contain these as subgroups.

[6]As for the Dirac-type phase, $\cos^2\delta$ can take a total of 34 different values associated with the sporadic patterns. However, note that in some cases the absolute values of the entries of the lepton mixing matrix do not depend on the Dirac phase δ. This happens when $\sin\theta_{12}\cos\theta_{12}\sin\theta_{23}\cos\theta_{13}\sin\theta_{13}=0$, in which case there is at least one null entry in U, and consequently there is no CP-violation associated to δ.

This leaves us with the infinite series of mixing patterns given in equation (10.1) as the only phenomenologically viable case; taking the 3 sigma range of $\sin^2\theta_{13}$ of Forero et al. [43], this translates into $-0.69 \lesssim \mathrm{Re}\left(\sigma^6\right) \lesssim -0.37$ for roots of unity σ. Suitable values of n include, sorted by group size,

- $n = 9, 18$ with $m = 18$, $G = (\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes S_3$ and $\mathrm{order}(G) = 648$,
- $n = 11, 22, 33, 66$ with $m = 22$, $G = \Delta(6 \times 22^2)$ and $\mathrm{order}(G) = 2904$,
- $n = 28, 84$ with $m = 28$, $G = \Delta(6 \times 28^2)$ and $\mathrm{order}(G) = 4704$,
- $n = 32, 96$ with $m = 32$, $G = \Delta(6 \times 32^2)$ and $\mathrm{order}(G) = 6144$,

among others. However, we note that in the past, with less precise neutrino oscillation data, values of n equal to 5 and 16 were valid as well — see for instance [21, 23, 37], see also [19, 20] for earlier references. One can easily check that equation (10.1) gives a trivial Dirac-type phase δ. In figure 1, the case $\mathcal{C}_2$ is represented by three lines which are obtained by varying the root of unity σ. The colours correspond to different column permutations:

- red: $\cos^2\theta_{13}\sin^2\theta_{12} = 1/3$,
- blue: $\cos^2\theta_{13}\cos^2\theta_{12} = 1/3$,
- green: $\sin^2\theta_{13} = 1/3$.

From the two lower plots in figure 1 we can read off that, if $\mathcal{C}_2$ is realized in nature, then $\sin^2\theta_{23}$ must be quite far from 0.5.

For completeness, we have also presented in appendix E the solutions where U is block-diagonal, i.e. one flavour does not mix with the other two. As a byproduct, our analysis has yielded all instances where one row of $|U|^2$ is fixed.[7] Actually, if one takes only the cases without a zero in the row, then there are only two cases, $\mathcal{P}_1$ and $\mathcal{P}_2$, which have for instance been found earlier in [24] with the help of GAP.

The result of our analysis is a bit sobering. Taking up the position that residual symmetries should be capable of reproducing the results of fitting U to the data [40–43], there is only the series $\mathcal{C}_2$ which can do the job; some mixing matrices of this series were discovered earlier in [21] by using GAP. For the sporadic mixing matrices, the main obstacle is to reproduce the small quantity $\sin^2\theta_{13} \simeq 0.021$ obtained from the oscillation data. On the one hand, some sporadic mixing matrices $|U|^2$ have a zero entry, which is ruled out as it would imply $\sin^2\theta_{13} = 0$. On the other hand, the smallest non-zero entry of all sporadic $|U|^2$ is $(5-\sqrt{21})/12 \simeq 0.035$ occurring in $\mathcal{C}_5$, which is significantly larger than the physical $\sin^2\theta_{13}$.

Maybe the result of this paper gives more credibility to models where mixing angles and phases are related to mass ratios. Such models can be based on Abelian and non-Abelian flavour groups. Texture zeros are one possibility — these are, in effect, practically synonymous with Abelian symmetries [38]. One might also try to relax the assumptions of

[7] Our analysis does *not* produce the cases where a column in U is determined.

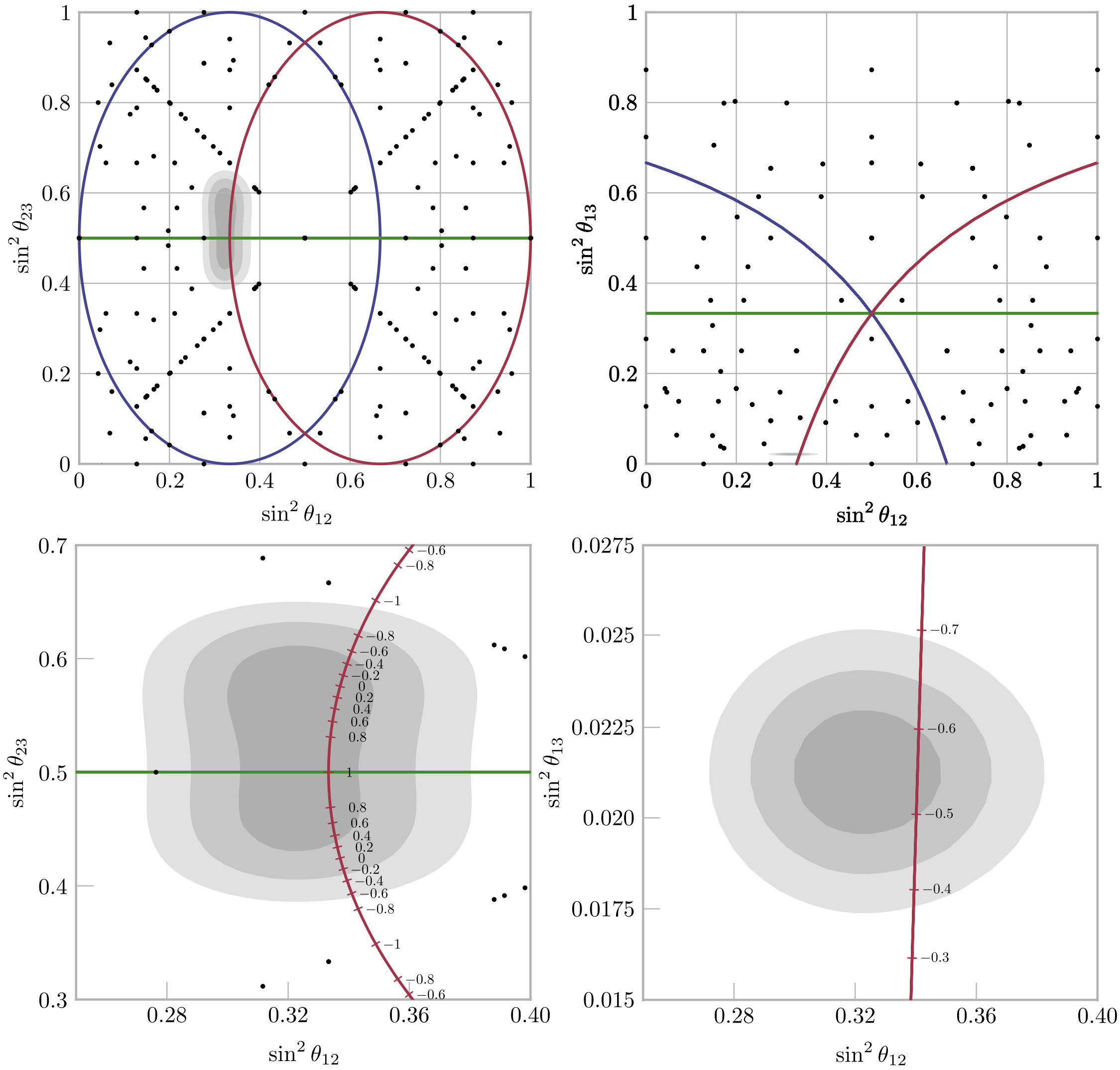

Figure 1. Plots with the values of the three lepton mixing angles in the standard parameterisation, for all possible mixing patterns involving genuine three-flavour mixing (including permutations). For each case pertaining to the sporadic mixing matrices, a dot has been placed in these plots (some of them are superimposed). The three lines (red, blue, green) correspond to different column permutations of the series of patterns obtained from equation (10.1) by varying the root of unity σ. The lower plots are magnifications of the ones above in the physically interesting region, with the numbers on the red curves indicating values of Re $\left(\sigma^6\right)$. The gray regions mark the 1, 2 and 3 sigma ranges of the mixing angles calculated by Forero et al. [43] for the normal mass hierarchy.

our analysis, for example allowing one neutrino mass to be zero [39], adopting the idea that neutrinos are Dirac particles,[8] or giving up the *ad hoc* assumption of a finite flavour group G. In particular, in the latter case a new mathematical approach would be necessary. It is also worth mentioning that, since neither of the residual symmetries (G_ℓ and G_ν) are symmetries of the full Lagrangian, in concrete models there will be radiative corrections to the mixing patterns presented here.

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A The possible eigenvalues of $Y^{(ij)}$

The relations in equation (3.5) have the generic form

$$\sum_{k=1}^{3} (\lambda_k + \lambda_k^*) + 2 = 0, \tag{A.1}$$

with roots of unity λ_1, λ_2, λ_3. This equation can be conceived as a vanishing sum over eight roots of unity such that the trivial root, 1, occurs in the sum and also the complex conjugate of every root.

A suitable theorem to deal with such a vanishing sum is theorem 6 of [30], which we have reproduced in this work as theorem 1 in section 2. This mathematical result deals with all vanishing sums of at most 9 roots of unity, which we have labeled from a) to h). In each sum, all roots are different. Let us now go through each of them. Sums g) and h) have each 9 roots, so these sums are irrelevant for a solution of equation (A.1). On the other hand, the sums e) and f) have 8 distinct roots of unity each, but in equation (A.1) the trivial root 1 occurs twice, so sums e) and f) are also irrelevant for our purpose. Sums c) and d) have 7 roots each, therefore, adding a further root of unity does not lead to a vanishing sum.

Sum b) has six roots, thus with this vanishing sum we can build a vanishing sum of eight roots of the form

$$\delta_1 \left(-\omega - \omega^2 + \beta + \beta^2 + \beta^3 + \beta^4\right) + \delta_2 \left(1 - 1\right) = 0, \tag{A.2}$$

[8] Note that, for neutrinos with Dirac nature, G_ν will in general not be the Klein four-group. However, if this is nontheless the case — see [27], then the analysis of this paper still applies.

where $\omega = e^{2\pi i/3}$, $\beta = e^{2\pi i/5}$, and δ_1 and δ_2 are further roots of unity yet to be specified. The trivial root 1 has to occur in both partial sums in equation (A.2). Therefore, without loss of generality, $\delta_2 = 1$. Let us first assume $-\delta_1\omega = 1$. Then equation (A.2) gives

$$1 + \omega - \omega^2 \left(\beta + \beta^2 + \beta^3 + \beta^4\right) + 1 - 1 = 0. \tag{A.3}$$

The root -1 should occur two times in this sum, but this is not the case and we can discard $-\delta_1\omega = 1$. We can deal similarly with $-\delta_1\omega^2 = 1$. Next we assume $\delta_1\beta = 1$, which leads to

$$-\omega\beta^4 - \omega^2\beta^4 + 1 + \beta + \beta^2 + \beta^3 + 1 - 1 = 0. \tag{A.4}$$

Again, -1 should occur twice, but it does not. Analogously, we can exclude all other possibilities like $\delta_1\beta^2 = 1$ and so on. In summary, we have found that sum b) of Conway and Jones cannot be part of the solution of equation (A.1).

With sum a) we have the vanishing sum of eight roots of unity

$$\delta_1 \left(1 + \beta + \beta^2 + \beta^3 + \beta^4\right) + \delta_2 \left(1 + \omega + \omega^2\right) = 0. \tag{A.5}$$

Since 1 must occur in both partial sums, we can choose without loss of generality $\delta_1 = \delta_2 = 1$. Indeed, this represents a solution of equation (A.1) with $\lambda_1 = \omega$, $\lambda_2 = \beta$, $\lambda_3 = \beta^2$.

According to the theorem of Conway and Jones, the remaining solutions of equation (A.1) can only contain powers of ω and -1. One possibility is

$$\delta_1 \left(1 + \omega + \omega^2\right) + \delta_2 \left(1 + \omega + \omega^2\right) + \delta_3 (1 - 1) = 0. \tag{A.6}$$

Suppose the trivial root 1 occurs in each of the partial sums with ω. Then, without loss of generality $\delta_1 = \delta_2 = 1$. This implies $\delta_3^* = -\delta_3$ or $\delta_3 = i$. Thus we have a second solution of equation (A.1): $\lambda_1 = i$, $\lambda_2 = \lambda_3 = \omega$. On the other hand, we can also assume that the trivial root 1 occurs in the first and the third partial sum of equation (A.6). Without loss of generality, this is achieved with $\delta_1 = \delta_3 = 1$. Since, in this case, -1 occurs in the third partial sum, it must occur in the second partial sum too. Without loss of generality, this implies $\delta_2 = -1$, leading to another solution $\lambda_1 = -1$, $\lambda_2 = \omega$, $\lambda_3 = -\omega$.

The last sum that needs to be discussed is of the form

$$\delta_1 (1 - 1) + \delta_2 (1 - 1) + \delta_3 (1 - 1) + \delta_4 (1 - 1) = 0. \tag{A.7}$$

Since the trivial root has to occur twice, we choose, without loss of generality, $\delta_1 = \delta_2 = 1$. Then $\delta_3 = \delta_4^* \equiv \lambda$ with an undetermined root λ. Thus we have found another solution of equation (A.1): $\lambda_1 = -1$, $\lambda_2 = \lambda$, $\lambda_3 = -\lambda$. The solution of the previous paragraph is special case of this one.

In summary, equation (A.1) has only three solutions, which are displayed in equation (3.11).

B The matrix elements $t_{ij} = |T_{ij}|^2$

Here we determine all possible forms of the matrix $t = (t_{ij})$. As we will see, it lies in the nature of the method that we can determine t only up to permutations, i.e. wherever we find a t, then also the matrices $P_1 t P_2$, where P_1 and P_2 are 3×3 permutation matrices, are admissible. In the present section, whenever we use "unique" for t, it is always meant unique up to such permutations.

According to equations (3.9) and (3.11), up to permutations, every line and column of t is identical to one of the three following possibilities:

$$\begin{array}{rl} \left(\frac{1}{2},\, \frac{1}{4},\, \frac{1}{4}\right) & \text{(A)}, \\ \left(\frac{1}{4},\, \frac{3+\sqrt{5}}{8},\, \frac{3-\sqrt{5}}{8}\right) & \text{(B)}, \\ \left(0,\, \frac{1+\cos\vartheta}{2},\, \frac{1-\cos\vartheta}{2}\right) & \text{(C)}. \end{array} \tag{B.1}$$

Note that

$$\left(\frac{\sqrt{5}\pm 1}{4}\right)^2 = \frac{3\pm\sqrt{5}}{8}. \tag{B.2}$$

The angle ϑ in (C) is an arbitrary rational angle,[9] but if we combine for instance a column of type (C) with a line of type (A) or (B), then $\cos\vartheta$ can assume only four values:

$$\begin{array}{rll} \cos\vartheta = 0: & \left(0,\, \frac{1}{2},\, \frac{1}{2}\right) & (\mathrm{C}_1), \\ \cos\vartheta = \frac{1}{2}: & \left(0,\, \frac{3}{4},\, \frac{1}{4}\right) & (\mathrm{C}_2), \\ \cos\vartheta = \frac{-1+\sqrt{5}}{4}: & \left(0,\, \frac{3+\sqrt{5}}{8},\, \frac{5-\sqrt{5}}{8}\right) & (\mathrm{C}_3), \\ \cos\vartheta = \frac{-1-\sqrt{5}}{4}: & \left(0,\, \frac{3-\sqrt{5}}{8},\, \frac{5+\sqrt{5}}{8}\right) & (\mathrm{C}_4). \end{array} \tag{B.3}$$

Later we will see in one instance that also the case

$$\cos\vartheta = 1: \quad (0,\, 1,\, 0) \quad (\mathrm{C}_0) \tag{B.4}$$

occurs.

In the following we derive the possible forms of t, by exhausting all possible combinations of (A), (B) and (C). To find all viable cases, we must take into account that t stems from a unitary matrix T. Firstly, an admissible t must fulfill

$$\sum_{k=1}^{3} t_{ik} = \sum_{k=1}^{3} t_{kj} = 1 \quad \forall\, i, j = 1, 2, 3. \tag{B.5}$$

Secondly, for such a t, one must also check that it is possible to find phases α_{ij} such that the scheme of complex numbers $\sqrt{t_{ij}}\, e^{i\alpha_{ij}}$ fulfills the orthogonality relations of a unitary matrix. It was shown in [33] that for this purpose it is sufficient to check the validity of the inequality

$$4 t_{11} t_{22} t_{12} t_{21} \geq \left(1 - t_{11} - t_{22} - t_{12} - t_{21} + t_{11} t_{22} + t_{12} t_{21}\right)^2. \tag{B.6}$$

B.1 First line in t of type (A)

Assuming the ordering of the first line as in equation (B.1), the first column can be either of type (A) or (C_1).

[9] Since ϑ comes from a root of unity, it must be 2π times some rational number, but otherwise it is arbitrary.

First column in t of type (A). Then the second line can be of type (A) or (B) or (C_2). If it is of type (A), we obtain

$$t = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad (\mathrm{AAA}). \tag{B.7}$$

This t is compatible with inequality (B.6). Here and in the following the capital letters to the right of the matrix indicate the types of the lines. If the second line is of type (B), we arrive at

$$t = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3+\sqrt{5}}{8} & \frac{3-\sqrt{5}}{8} \\ \frac{1}{4} & \frac{3-\sqrt{5}}{8} & \frac{3+\sqrt{5}}{8} \end{pmatrix} \quad (\mathrm{ABB}). \tag{B.8}$$

This case is also compatible with inequality (B.6). Finally, if the second line is of type (C_2), we find

$$t = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \quad (\mathrm{AC_2C_2}). \tag{B.9}$$

This case is *not* in agreement with inequality (B.6).

First column in t of type (C_1). Let us assume $t_{21} = 1/2$ and $t_{31} = 0$. Then the second line can be of type (A) or (C_1). The first possibility gives, upon reordering of the lines,

$$t = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \quad (\mathrm{C_1AA}). \tag{B.10}$$

This is a viable case with respect to inequality (B.6). If we have type (C_1) in the second line, then the third line must be (C_2):

$$t = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix} \quad (\mathrm{AC_1C_2}). \tag{B.11}$$

However, this case does not comply with inequality (B.6). Now we have exhausted all possibilities with one or more lines of type (A).

B.2 First line in t of type (B), but no line of type (A)

Assuming the ordering of (B) as in equation (B.1), the first column can be either of type (A) or (B) or (C_2).

First column of type (A). With the ordering of the first column as $(1/4, 1/4, 1/2)$, the third line must be either of type (A) or of type (C_1). If it is of type (A), then the second line must be of type (B) and we are lead to a permutation of (ABB) of equation (B.8). If it is of type (C_1), then the column which has the zero must be of type (C_3) or (C_4). But then the second line cannot be one of the types (A), (B), (C). So we come to the conclusion that, if the first column is of type (A), no new case ensues.

First column of type (B). With $t_{21} = (3 - \sqrt{5})/8$, the second line can be (B) or (C_4). In the first case we are lead to the viable matrix

$$t = \begin{pmatrix} \frac{1}{4} & \frac{3+\sqrt{5}}{8} & \frac{3-\sqrt{5}}{8} \\ \frac{3-\sqrt{5}}{8} & \frac{1}{4} & \frac{3+\sqrt{5}}{8} \\ \frac{3+\sqrt{5}}{8} & \frac{3-\sqrt{5}}{8} & \frac{1}{4} \end{pmatrix} \quad \text{(BBB)}. \tag{B.12}$$

If the second line is of type (C_4), the only way to fulfill equation (B.5) is

$$t = \begin{pmatrix} \frac{1}{4} & \frac{3+\sqrt{5}}{8} & \frac{3-\sqrt{5}}{8} \\ \frac{3-\sqrt{5}}{8} & 0 & \frac{5+\sqrt{5}}{8} \\ \frac{3+\sqrt{5}}{8} & \frac{5-\sqrt{5}}{8} & 0 \end{pmatrix} \quad (\mathrm{BC_4C_3}). \tag{B.13}$$

However, this t is not admissible because inequality (B.6) is not fulfilled.

First column of type (C_2). Assuming $t_{21} = 3/4$, then the second line must be of type (C_2). In this case we find

$$t = \begin{pmatrix} \frac{1}{4} & \frac{3+\sqrt{5}}{8} & \frac{3-\sqrt{5}}{8} \\ \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{3-\sqrt{5}}{8} & \frac{5+\sqrt{5}}{8} \end{pmatrix} \quad (\mathrm{BC_2C_4}). \tag{B.14}$$

and

$$t = \begin{pmatrix} \frac{1}{4} & \frac{3+\sqrt{5}}{8} & \frac{3-\sqrt{5}}{8} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & \frac{5-\sqrt{5}}{8} & \frac{3+\sqrt{5}}{8} \end{pmatrix} \quad (\mathrm{BC_2C_3}). \tag{B.15}$$

However, both possibilities must be discarded because inequality (B.6) is not fulfilled. Now we have exhausted all cases with lines of type (A) and (B). It remains to discuss matrices t where *all* lines are of type (C).

B.3 All lines of type (C)

Let us assume that the first line is of type (C_0). Then we find

$$t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1+\cos\vartheta}{2} & \frac{1-\cos\vartheta}{2} \\ 0 & \frac{1-\cos\vartheta}{2} & \frac{1+\cos\vartheta}{2} \end{pmatrix} \quad (\mathrm{C_0CC}). \tag{B.16}$$

All $\cos\vartheta$ are in agreement with inequality (B.6). If no line of type (C_0) occurs in t, i.e. $\cos\vartheta \neq \pm 1$, then one quickly finds that up to permutations the unique possibility is

$$t = \begin{pmatrix} 0 & \frac{1+\cos\vartheta}{2} & \frac{1-\cos\vartheta}{2} \\ \frac{1-\cos\vartheta}{2} & 0 & \frac{1+\cos\vartheta}{2} \\ \frac{1+\cos\vartheta}{2} & \frac{1-\cos\vartheta}{2} & 0 \end{pmatrix} \quad \text{(CCC)}. \tag{B.17}$$

However, application of inequality (B.6) gives $\cos^2\vartheta = 1$, which we had already excluded.

In summary, equation (B.1), which we had derived using the theorem of Conway and Jones, leads to 11 matrices t which comply with equation (B.5). However, checking these matrices against inequality (B.6), we are left with only five viable cases:

$$\text{(AAA)},\ \text{(ABB)},\ (\text{C}_1\text{AA}),\ \text{(BBB)},\ (\text{C}_0\text{CC}).$$

In the latter case, the angle ϑ is not restricted, apart from being a rational angle.

C Details of the derivation of the external phases

C.1 Form 2A

According to the procedure explained in the introduction to section 8, we first compute

$$T^2 = \begin{pmatrix} \frac{1}{2}(\kappa_2+\kappa_3) & -\frac{1}{2\sqrt{2}}(\kappa_2-\kappa_3) & \frac{1}{2\sqrt{2}}(\kappa_2-\kappa_3) \\ -\frac{1}{2\sqrt{2}}(\kappa_2-\kappa_3) & \frac{1}{4}(2\kappa_1+\kappa_2+\kappa_3) & \frac{1}{4}(2\kappa_1-\kappa_2-\kappa_3) \\ \frac{1}{2\sqrt{2}}(\kappa_2-\kappa_3) & \frac{1}{4}(2\kappa_1-\kappa_2-\kappa_3) & \frac{1}{4}(2\kappa_1+\kappa_2+\kappa_3) \end{pmatrix} \hat{\kappa}. \tag{C.1}$$

Since $T^2 \in G$ and having excluded the basic forms 1 and 4, we know that $|T^2|$ has to be one of the basic forms 2, 3 or 5. The $|T^2|$ above follows the pattern

$$\left|T^2\right| = \begin{pmatrix} A & B & B \\ B & D & C \\ B & C & D \end{pmatrix}. \tag{C.2}$$

Since B occurs four times in $|T^2|$, it follows from equation (3.12) that $B = 0$, $1/\sqrt{2}$ or $1/2$, corresponding to

$$\kappa_2 = \kappa_3, \quad \kappa_2 = -\kappa_3, \quad \text{or} \quad \kappa_2 = \pm i\kappa_3, \tag{C.3}$$

respectively. In the latter case one finds $A = 1/\sqrt{2}$, so there could be one, three or five elements $1/\sqrt{2}$ in $|T^2|$; however, a glance at equation (3.12) reveals that this is impossible, because $1/\sqrt{2}$ appears repeated an even number of times in the basic forms 2, 3 and 5. Therefore, the viable possibilities are $\kappa_2 = \pm\kappa_3$.

C.2 Form 2B

Proceeding in the same way as in the previous subsection, we find

$$T^2 = \begin{pmatrix} \frac{\kappa_1}{2} + \frac{\kappa_3}{2\sqrt{2}} & \frac{\kappa_3}{2} & \frac{\kappa_1}{2} - \frac{\kappa_3}{2\sqrt{2}} \\ -\frac{\kappa_1}{2\sqrt{2}} - \frac{\kappa_2}{2\sqrt{2}} + \frac{\kappa_3}{4} & \frac{\kappa_2}{2} + \frac{\kappa_3}{2\sqrt{2}} & -\frac{\kappa_1}{2\sqrt{2}} + \frac{\kappa_2}{2\sqrt{2}} - \frac{\kappa_3}{4} \\ \frac{\kappa_1}{2\sqrt{2}} - \frac{\kappa_2}{2\sqrt{2}} - \frac{\kappa_3}{4} & \frac{\kappa_2}{2} - \frac{\kappa_3}{2\sqrt{2}} & \frac{\kappa_1}{2\sqrt{2}} + \frac{\kappa_2}{2\sqrt{2}} + \frac{\kappa_3}{4} \end{pmatrix} \hat{\kappa}, \tag{C.4}$$

and $\left|T^2\right|$ follows the pattern

$$\left|T^2\right| = \begin{pmatrix} A & \frac{1}{2} & B \\ E & C & F \\ G & D & H \end{pmatrix}. \tag{C.5}$$

Because of

$$\left|\left(T^2\right)_{ij}\right| \geq \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) \quad \text{for} \quad (i,j) = (1,1),\ (1,3),\ (2,2),\ (3,2), \tag{C.6}$$

there are, together with $\left|\left(T^2\right)_{12}\right|$, at least five non-zero elements in $\left|T^2\right|$. Moreover, of the remaining four elements, not all can be zero at the same time. Thus there are more than five non-zero elements in $\left|T^2\right|$, which excludes the basic form 5. Thus only the basic forms 2 and 3 with all possible permutations come into question, with the boundary condition $\left|\left(T^2\right)_{12}\right| = 1/2$. First of all, this gives

$$A = \frac{1}{\sqrt{2}},\ \frac{1}{2},\ \frac{\sqrt{5}+1}{4} \text{ or } \frac{\sqrt{5}-1}{4}. \tag{C.7}$$

Secondly, there are eight possible forms of $\left|T^2\right|$ which are

$$\begin{aligned}
&\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, &&\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}, \\
&\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, &&\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \\
&\begin{pmatrix} \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \end{pmatrix}, &&\begin{pmatrix} \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix}, \\
&\begin{pmatrix} \frac{\sqrt{5}-1}{4} & \frac{1}{2} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \end{pmatrix}, &&\begin{pmatrix} \frac{\sqrt{5}-1}{4} & \frac{1}{2} & \frac{\sqrt{5}+1}{4} \\ \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} & \frac{1}{2} \end{pmatrix}.
\end{aligned} \tag{C.8}$$

Considering these matrices, we see that always one of the following two equalities holds:

$$A = C \quad \text{or} \quad A = D. \tag{C.9}$$

Exploiting these equations, we obtain

$$\left|\frac{\kappa_1}{2} + \frac{\kappa_3}{2\sqrt{2}}\right| = \left|\frac{\kappa_2}{2} \pm \frac{\kappa_3}{2\sqrt{2}}\right| \quad \Rightarrow \quad \mathrm{Re}\left(\kappa_1\kappa_3^*\right) = \pm\mathrm{Re}\left(\kappa_2\kappa_3^*\right). \tag{C.10}$$

Defining $x = \kappa_1\kappa_3^*$, the external phases are given by

$$\hat{\kappa} = (x, \pm x, 1)\,\kappa_3 \quad \text{or} \quad (x, \pm x^*, 1)\,\kappa_3. \tag{C.11}$$

However, the first relation leads to $\left|(T^2)_{21}\right| = 1/4$ or $\left|(T^2)_{31}\right| = 1/4$, which cannot occur in the basic forms 2 and 3. This leaves us with

$$A = C: \quad \hat{\kappa} = (x, x^*, 1)\,\kappa_3 \quad \text{or} \quad A = D: \quad \hat{\kappa} = (x, -x^*, 1)\,\kappa_3. \tag{C.12}$$

Finally, we are in a position to determine x for all cases of A:

$$A = \left|\frac{x}{2} + \frac{1}{2\sqrt{2}}\right| \quad \Rightarrow \quad \begin{cases} x = -\varphi & \text{for } A = \frac{1}{2}, \\ x = \varphi & \text{for } A = \frac{1}{\sqrt{2}}, \\ x = \rho_0 & \text{for } A = \frac{\sqrt{5}+1}{4}, \\ x = -\rho_0 & \text{for } A = \frac{\sqrt{5}-1}{4}. \end{cases} \tag{C.13}$$

The phase factors φ and ρ_0 are defined in equation (6.4). Of course, for every x also x^* is a solution of equation (C.13), but in the light of the discussion in section 4 this is a trivial variation.

C.3 Form 3A

As before, we consider T^2, which in the present case is given by

$$T^2 = \begin{pmatrix} \frac{1}{4}(\kappa_1 - \kappa_2 - \kappa_3) & \frac{1}{8}\left[-a(\kappa_1 + \kappa_2) + b\kappa_3\right] & \frac{1}{8}\left[-c(\kappa_1 + \kappa_3) - d\kappa_2\right] \\ \frac{1}{8}\left[c(\kappa_1 + \kappa_2) + d\kappa_3\right] & \frac{1}{4}(-\kappa_1 + \kappa_2 - \kappa_3) & \frac{1}{8}\left[a(\kappa_2 + \kappa_3) - b\kappa_1\right] \\ \frac{1}{8}\left[a(\kappa_1 + \kappa_3) - b\kappa_2\right] & \frac{1}{8}\left[-d\kappa_1 - c(\kappa_2 + \kappa_3)\right] & \frac{1}{4}(-\kappa_1 - \kappa_2 + \kappa_3) \end{pmatrix} \hat{\kappa}, \tag{C.14}$$

where we have defined

$$a = -1 + \sqrt{5}, \quad b = 3 + \sqrt{5}, \quad c = 1 + \sqrt{5}, \quad d = 3 - \sqrt{5}. \tag{C.15}$$

It is easy to show that none of the entries of $|T^2|$ can be as large as one, which rules out form 5. So it remains to consider forms 2 and 3. Another observation is that phase factors cannot be "aligned," i.e. $\kappa_j = \pm\kappa_k$ $(j \neq k)$, because otherwise one or more entries at the diagonal of $|T^2|$ would be $1/4$ which is forbidden for all basic forms except form 5 which we have excluded before.

Let us first envisage that $|T^2|$ is of form 3. A specific property of form 3 is that either all elements on the diagonal are the same or all three elements are different. The latter case is impossible because $\left|(T^2)_{jj}\right| \leq 3/4$, while one element on the diagonal would have to be $(\sqrt{5}+1)/4$ which is larger than $3/4$. Since we know that all elements are the same on the diagonal, we deduce

$$\begin{aligned} \left|(T^2)_{11}\right| = \left|(T^2)_{22}\right| &\Rightarrow \mathrm{Re}\,(\kappa_1\kappa_3^*) = \mathrm{Re}\,(\kappa_2\kappa_3^*)\,, \\ \left|(T^2)_{22}\right| = \left|(T^2)_{33}\right| &\Rightarrow \mathrm{Re}\,(\kappa_2\kappa_1^*) = \mathrm{Re}\,(\kappa_3\kappa_1^*)\,. \end{aligned} \tag{C.16}$$

Thus we conclude that either $\kappa_1\kappa_3^* = \kappa_2\kappa_3^*$ or $\kappa_1\kappa_3^* = \kappa_2^*\kappa_3$. In the first case we would have alignment of phase factors, which we have excluded before. Therefore, equation (C.16) leads to the relations

$$\kappa_1\kappa_2 = \kappa_3^2, \quad \kappa_2\kappa_3 = \kappa_1^2. \tag{C.17}$$

These relations are solved by $\hat{\kappa} = \mathrm{diag}\,(1, \omega^k, \omega^{2k})\kappa_1$ with $k = 1, 2$, since $k = 0$ would lead to alignment of phase factors. Since κ_1 is an arbitrary root of unity and complex conjugation of T leads to a trivial variation, we obtain without loss of generality

$$\hat{\kappa} = \mathrm{diag}\,(1, \omega, \omega^2)\kappa_1 \tag{C.18}$$

when $|T^2|$ is of form 3. All elements on the diagonal of $|T^2|$ are $1/2$ in this case.

Next we discuss the possibility that $|T^2|$ is of form 2, in which case there must be one zero in the matrix. Because of $b > 2a$, we have

$$\left|\left(T^2\right)_{ij}\right| > 0 \quad \text{for} \quad (i,j) = (1,2),\ (2,3),\ (3,1). \tag{C.19}$$

We further observe that

$$\left(T^2\right)_{ij} = 0 \;\Rightarrow\; \left|\left(T^2\right)_{ji}\right| = \frac{ad+cb}{8c} = \frac{5-\sqrt{5}}{4}, \quad \text{if} \quad (i,j) = (2,1),\ (3,2),\ \text{or}\ (1,3), \tag{C.20}$$

which is impossible because no such element occurs in form 2. We conclude that the zero must be on the diagonal of T^2:

- $\left(T^2\right)_{11} = 0 \Rightarrow \kappa_1 = \kappa_2 + \kappa_3 \Rightarrow |\kappa_2 + \kappa_3| = 1$. This implies $\kappa_2\kappa_3^* = \omega$ or ω^2, so we end up with $\hat{\kappa} = \mathrm{diag}\,\left(-\omega^2, \omega, 1\right)\kappa_3$ or $\mathrm{diag}\,\left(-\omega, \omega^2, 1\right)\kappa_3$.
- $\left(T^2\right)_{22} = 0 \Rightarrow \kappa_2 = \kappa_1 + \kappa_3 \Rightarrow |\kappa_1 + \kappa_3| = 1$. This implies $\kappa_1\kappa_3^* = \omega$ or ω^2, so we end up with $\hat{\kappa} = \mathrm{diag}\,\left(\omega, -\omega^2, 1\right)\kappa_3$ or $\mathrm{diag}\,\left(\omega^2, -\omega, 1\right)\kappa_3$.
- $\left(T^2\right)_{33} = 0 \Rightarrow \kappa_3 = \kappa_1 + \kappa_2 \Rightarrow |\kappa_1 + \kappa_2| = 1$. This implies $\kappa_1\kappa_2^* = \omega$ or ω^2, so we end up with $\hat{\kappa} = \mathrm{diag}\,\left(\omega, 1, -\omega^2\right)\kappa_2$ or $\mathrm{diag}\,\left(\omega^2, 1, -\omega\right)\kappa_2$.

Leaving out cases which emerge from each other by complex conjugation, we end up with the three solutions

$$\hat{\kappa} = \mathrm{diag}\,\left(-\omega^2, \omega, 1\right)\kappa_3,\ \mathrm{diag}\,\left(\omega, -\omega^2, 1\right)\kappa_3,\ \mathrm{diag}\,\left(\omega, 1, -\omega^2\right)\kappa_2 \tag{C.21}$$

for the external phases, where κ_2 and κ_3 are arbitrary roots of unity.

C.4 Form 3B

The matrix T^2 is as follows:

$$T^2 = \begin{pmatrix} \frac{1}{8}\left(2\kappa_1 - b\kappa_2 - d\kappa_3\right) & -\frac{1}{8}\left(c\kappa_1 + 2\kappa_2 + a\kappa_3\right) & -\frac{1}{8}\left(a\kappa_1 + c\kappa_2 - 2\kappa_3\right) \\ \frac{1}{8}\left(c\kappa_1 + 2\kappa_2 + a\kappa_3\right) & \frac{1}{8}\left(-b\kappa_1 + d\kappa_2 + 2\kappa_3\right) & \frac{1}{8}\left(-2\kappa_1 + a\kappa_2 - c\kappa_3\right) \\ \frac{1}{8}\left(a\kappa_1 + c\kappa_2 - 2\kappa_3\right) & \frac{1}{8}\left(-2\kappa_1 + a\kappa_2 - c\kappa_3\right) & \frac{1}{8}\left(-d\kappa_1 + 2\kappa_2 + b\kappa_3\right) \end{pmatrix} \hat{\kappa}, \tag{C.22}$$

where a, b, c, d are defined in equation (C.15). The most obvious observation is that $\left|T^2\right|$ is symmetric. Therefore, if $|T^2|$ is if form 2 which has one zero, this zero can only be on the diagonal. However, $b > 2 + d$ rules out this possibility. So it remains to consider $|T^2|$ being of form 3 and of form 5.

If $|T^2|$ is of form 3, then the elements on the diagonal are either all the same or all different. Since the first option leads to a non-symmetric matrix, we are left with the second option, namely having 1/2, $(\sqrt{5}+1)/4$ and $(\sqrt{5}-1)/4$ on the diagonal. Taking into account that

$$\left|\left(T^2\right)_{jj}\right| \geq \frac{1}{8}(b-2-d) = \frac{\sqrt{5}-1}{4}, \tag{C.23}$$

we have three cases:

- $\left|\left(T^2\right)_{11}\right| = (\sqrt{5}-1)/4 \Rightarrow \hat{\kappa} = \text{diag}\,(1,1,-1)\,\kappa_1$,
- $\left|\left(T^2\right)_{22}\right| = (\sqrt{5}-1)/4 \Rightarrow \hat{\kappa} = \text{diag}\,(1,1,1)\,\kappa_1$,
- $\left|\left(T^2\right)_{33}\right| = (\sqrt{5}-1)/4 \Rightarrow \hat{\kappa} = \text{diag}\,(1,-1,1)\,\kappa_1$.

If $|T^2|$ is of form 5, we observe that

$$\left|\left(T^2\right)_{ij}\right| \leq \frac{1}{8}(2+a+c) = \frac{\sqrt{5}+1}{4} < 1 \quad \forall\, i \neq j. \tag{C.24}$$

Therefore, the 1 has to be somewhere on the diagonal. Since

$$\frac{1}{8}(2+b+d) = 1 \tag{C.25}$$

holds, the phase factors necessarily fulfill $\kappa_2 = \kappa_3 = -\kappa_1$, leading to $\left|T^2\right| = \mathbb{1}$, consistent with $2+a-c=0$ for the off-diagonal elements. In summary, the external phases are given by

$$\hat{\kappa} = (-1,1,1)\,\kappa_3. \tag{C.26}$$

C.5 Form 3C

This case is quite similar to form 3A. With

$$T^2 = \begin{pmatrix} \frac{1}{8}\left[b\kappa_1 + a\left(\kappa_2-\kappa_3\right)\right] & \frac{1}{8}\left[c\left(\kappa_2-\kappa_1\right) - d\kappa_3\right] & \frac{1}{4}\left(\kappa_1+\kappa_2+\kappa_3\right) \\ \frac{1}{4}\left(-\kappa_1+\kappa_2+\kappa_3\right) & \frac{1}{8}\left[b\kappa_2 + a\left(\kappa_1+\kappa_3\right)\right] & \frac{1}{8}\left[c\left(\kappa_2-\kappa_3\right) - d\kappa_1\right] \\ \frac{1}{8}\left[-c\left(\kappa_1+\kappa_3\right) + d\kappa_2\right] & \frac{1}{4}\left(\kappa_1+\kappa_2-\kappa_3\right) & \frac{1}{8}\left[b\kappa_3 + a\left(\kappa_2-\kappa_1\right)\right] \end{pmatrix} \hat{\kappa} \tag{C.27}$$

and a, b, c, d defined in equation (C.15), we find that none of its entries can be as large as 1, which rules out form 5. Furthermore, the phase factors κ_j cannot be aligned, lest one or more entries of $|T^2|$ be 1/4.

First we consider form 3. The entries

$$\left|\left(T^2\right)_{ij}\right| \quad \text{with} \quad (i,j) = (1,3),\ (2,1),\ (3,2) \tag{C.28}$$

are either all the same or all different. However, the latter case cannot occur, because these entries are all smaller or equal 3/4, which is smaller than $(\sqrt{5}+1)/4$. Evaluating the requirement that the elements of equation (C.28) have to be equal, we proceed along the same lines as for form 3A and obtain the relations

$$\kappa_1^2 = -\kappa_2\kappa_3, \quad \kappa_3^2 = -\kappa_1\kappa_2. \tag{C.29}$$

From this we arrive at the external phases

$$\hat{\kappa} = \left(\omega, -\omega^2, 1\right) \kappa_3. \tag{C.30}$$

It remains to consider form 2, which has one zero in some place. The zero cannot be on the diagonal of $|T^2|$ because $b - 2a = 5 - \sqrt{5} > 0$. Moreover, proceeding as in equation (C.20), we find

- $\left(T^2\right)_{12} = 0 \Rightarrow \left|\left(T^2\right)_{33}\right| = (5 - \sqrt{5})/4,$
- $\left(T^2\right)_{23} = 0 \Rightarrow \left|\left(T^2\right)_{11}\right| = (5 - \sqrt{5})/4,$
- $\left(T^2\right)_{31} = 0 \Rightarrow \left|\left(T^2\right)_{22}\right| = (5 - \sqrt{5})/4,$

which is impossible for form 2. Thus we have the following three possibilities:

- $\left(T^2\right)_{21} = 0 \Rightarrow \kappa_1 = \kappa_2 + \kappa_3 \Rightarrow |\kappa_2 + \kappa_3| = 1$. This implies $\kappa_2 \kappa_3^* = \omega$ or ω^2, so we end up with $\hat{\kappa} = \operatorname{diag}\left(-\omega^2, \omega, 1\right) \kappa_3$ or $\operatorname{diag}\left(-\omega, \omega^2, 1\right) \kappa_3$.
- $\left(T^2\right)_{13} = 0 \Rightarrow \kappa_2 = -\kappa_1 - \kappa_3 \Rightarrow |\kappa_1 + \kappa_3| = 1$. This implies $\kappa_1 \kappa_3^* = \omega$ or ω^2, so we end up with $\hat{\kappa} = \operatorname{diag}\left(\omega, \omega^2, 1\right) \kappa_3$ or $\operatorname{diag}\left(\omega^2, \omega, 1\right) \kappa_3$.
- $\left(T^2\right)_{32} = 0 \Rightarrow \kappa_3 = \kappa_1 + \kappa_2 \Rightarrow |\kappa_1 + \kappa_2| = 1$. This implies $\kappa_1 \kappa_2^* = \omega$ or ω^2, so we end up with $\hat{\kappa} = \operatorname{diag}\left(\omega, 1, -\omega^2\right) \kappa_2$ or $\operatorname{diag}\left(\omega^2, 1, -\omega\right) \kappa_2$.

Leaving out the complex conjugate solutions, we finally obtain

$$\hat{\kappa} = \operatorname{diag}\left(-\omega^2, \omega, 1\right) \kappa_3,\ \operatorname{diag}\left(\omega, \omega^2, 1\right) \kappa_3,\ \operatorname{diag}\left(\omega, 1, -\omega^2\right) \kappa_2 \tag{C.31}$$

for the external phases, with κ_2 and κ_3 being arbitrary roots of unity.

C.6 Form 3D

Also this case is treated analogously to form 3A. With

$$T^2 = \begin{pmatrix} \frac{1}{8}\left[d\kappa_1 - c\left(\kappa_2 + \kappa_3\right)\right] & \frac{1}{4}\left(\kappa_1 - \kappa_2 + \kappa_3\right) & \frac{1}{8}\left[a\left(\kappa_3 - \kappa_1\right) - b\kappa_2\right] \\ \frac{1}{8}\left[a\left(\kappa_2 - \kappa_1\right) - b\kappa_3\right] & \frac{1}{8}\left[c\left(\kappa_3 - \kappa_1\right) + d\kappa_2\right] & \frac{1}{4}\left(\kappa_1 + \kappa_2 + \kappa_3\right) \\ \frac{1}{4}\left(\kappa_1 + \kappa_2 - \kappa_3\right) & \frac{1}{8}\left[b\kappa_1 + a\left(\kappa_2 + \kappa_3\right)\right] & \frac{1}{8}\left[c\left(\kappa_2 - \kappa_1\right) + d\kappa_3\right] \end{pmatrix} \hat{\kappa} \tag{C.32}$$

and a, b, c, d defined in equation (C.15), we find again that none of its entries can be as large as one, thus ruling out form 5. Furthermore, the phase factors κ_j cannot be aligned in order to avoid one or more entries of $|T^2|$ being $1/4$.

First we consider form 3. The entries

$$\left|\left(T^2\right)_{ij}\right| \quad \text{with} \quad (i,j) = (1,2),\ (2,3),\ (3,1) \tag{C.33}$$

are either all the same or all different. However, the latter case cannot occur, because $3/4$ is an upper bound to these entries and $3/4 < (\sqrt{5}+1)/4$. Evaluating the requirement that the elements of equation (C.33) have to be equal, we obtain the relations

$$\kappa_2^2 = -\kappa_1\kappa_3, \quad \kappa_3^2 = -\kappa_1\kappa_2, \tag{C.34}$$

which give

$$\hat{\kappa} = \left(-\omega, \omega^2, 1\right) \kappa_3. \tag{C.35}$$

Assuming now that $|T^2|$ is of form 2, there must be a zero in the matrix. However,

$$\left|\left(T^2\right)_{ij}\right| > 0 \quad \text{for} \quad (i,j) = (2,1),\ (3,2),\ (1,3) \tag{C.36}$$

because of $b > 2a$. Moreover, it follows that

- $\left(T^2\right)_{11} = 0 \Rightarrow \left|\left(T^2\right)_{32}\right| = (5-\sqrt{5})/4$,
- $\left(T^2\right)_{22} = 0 \Rightarrow \left|\left(T^2\right)_{13}\right| = (5-\sqrt{5})/4$,
- $\left(T^2\right)_{33} = 0 \Rightarrow \left|\left(T^2\right)_{21}\right| = (5-\sqrt{5})/4$,

in the same vein as for form 3A in equation (C.20), which is impossible because no such element occurs in form 2. Thus we have to consider the remaining three possibilities:

- $\left(T^2\right)_{12} = 0 \Rightarrow \kappa_2 = \kappa_1 + \kappa_3 \Rightarrow |\kappa_1 + \kappa_3| = 1$. This implies $\kappa_1 \kappa_3^* = \omega$ or ω^2, so we end up with $\hat{\kappa} = \operatorname{diag}\left(\omega, -\omega^2, 1\right) \kappa_3$ or $\operatorname{diag}\left(\omega^2, -\omega, 1\right) \kappa_3$.
- $\left(T^2\right)_{23} = 0 \Rightarrow \kappa_2 = -\kappa_1 - \kappa_3 \Rightarrow |\kappa_1 + \kappa_3| = 1$. This implies $\kappa_1 \kappa_3^* = \omega$ or ω^2, so we end up with $\hat{\kappa} = \operatorname{diag}\left(\omega, \omega^2, 1\right) \kappa_3$ or $\operatorname{diag}\left(\omega^2, \omega, 1\right) \kappa_3$.
- $\left(T^2\right)_{31} = 0 \Rightarrow \kappa_3 = \kappa_1 + \kappa_2 \Rightarrow |\kappa_1 + \kappa_2| = 1$. This implies $\kappa_1 \kappa_2^* = \omega$ or ω^2, so we end up with $\hat{\kappa} = \operatorname{diag}\left(\omega, 1, -\omega^2\right) \kappa_2$ or $\operatorname{diag}\left(\omega^2, 1, -\omega\right) \kappa_2$.

Dropping the complex conjugate solutions, we arrive at

$$\hat{\kappa} = \operatorname{diag}\left(\omega, -\omega^2, 1\right) \kappa_3,\ \operatorname{diag}\left(\omega, \omega^2, 1\right) \kappa_3,\ \operatorname{diag}\left(\omega, 1, -\omega^2\right) \kappa_2, \tag{C.37}$$

with κ_2 and κ_3 being arbitrary roots of unity.

D Minimal groups for $\mathcal{C}_2$

Here we determine minimal flavour groups G associated with the series $\mathcal{C}_2$. The generators of the groups of this series are

$$T = \kappa_1 \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \operatorname{diag}\left(1, \sigma^{-3}, -\sigma^{-3}\right) \tag{D.1}$$

and S_j $(j = 1,2,3)$. We make a basis transformation with

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}}\sigma^{-2} & -\frac{1}{\sqrt{2}}\sigma^{-2} \\ 0 & -\frac{1}{\sqrt{2}}\sigma^{-1} & \frac{1}{\sqrt{2}}\sigma^{-1} \end{pmatrix}, \tag{D.2}$$

leading to

$$T' = VTV^\dagger = \kappa E^2 \quad \text{with} \quad E^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{D.3}$$

and $S'_j = V S_j V^\dagger$ given by

$$S'_1 = S_1, \quad S'_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \sigma^* \\ 0 & \sigma & 0 \end{pmatrix}, \quad S'_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\sigma^* \\ 0 & -\sigma & 0 \end{pmatrix}. \tag{D.4}$$

For the time being, we set $\kappa = 1$, which entails $T' = E^2$ and $\sigma = -\kappa_2^*$. At any rate, σ is an abitrary root of unity. It is now useful to switch to another set of generators, as described in appendix F of [5]. For this purpose we compute

$$E \left(E S'_3\right)^2 E^\dagger = \text{diag}\left(\sigma^2, \sigma^*, \sigma^*\right) \equiv \tilde{F} \in G. \tag{D.5}$$

Instead of E it is certainly admissible to use $\tilde{E} \equiv \tilde{F} E$ and $\tilde{F}$ as generators. But then, with a further basis change, one can remove the phases from $\tilde{E}$ and S'_3:

$$V_1 = \text{diag}\left(\sigma^*, \sigma, 1\right) \tag{D.6}$$

and

$$V_1 \tilde{E} V_1^\dagger = E, \quad V_1 S'_3 V_1^\dagger \equiv B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{D.7}$$

In this final basis, our group G has the generators S_1, $\tilde{F}$, E and B. Clearly, E and B are generators of permutations, so G has the structure [5] $G = N \rtimes S_3$, where N is the normal subgroup consisting of all diagonal matrices and S_3 consists of all 3×3 permutations matrices. This N is generated by S_1, $\tilde{F}$, and matrices where the elements on the diagonal of S_1 and $\tilde{F}$ are permuted.

We can write σ as $\sigma = e^{2\pi i p/n}$ with integers p and n, and p coprime to n. If n is even, we define

$$E(\tilde{F})^{-1} E^\dagger = \text{diag}\left(\sigma, \sigma, \sigma^{-2}\right) \equiv F_1. \tag{D.8}$$

If n is odd, we consider $-\sigma$ instead for the definition of F_1. In this case we have

$$-\sigma = \exp\left(2\pi i \frac{2p+n}{2n}\right). \tag{D.9}$$

Thus it is useful to define $n' = \text{lcm}(2, n)$ and p', where $p' = p$ for n even and $p' = 2p + n$ for n odd. Note that p' is coprime to n'. Since

$$F_1^{n'/2} = \text{diag}\left(-1, -1, 1\right) = E^\dagger S_1 E, \tag{D.10}$$

we can dispense with the generator S_1.

According to theorem 1 in [44], there is a procedure to find two generators which reveal the structure of N and, as a consequence, the structure of G. We first need to find the element with the highest order in N. Since p' is coprime to n', there must be a positive integer a such that

$$F \equiv F_1^a = \text{diag}\left(\epsilon, \epsilon, \epsilon^{-2}\right) \quad \text{with} \quad \epsilon = e^{2\pi i/n'}. \tag{D.11}$$

This matrix has the highest possible order in N. In the next step we have to find the generator of the subgroup of N which has 1 as the first entry. It is easy to see that in our case every element in this subgroup is a power of

$$g \equiv \text{diag}\left(1, \epsilon^{-3}, \epsilon^3\right). \tag{D.12}$$

If $3 \nmid n'$, there is a positive integer b such that

$$g^b = \text{diag}\left(1, \epsilon^*, \epsilon\right) \tag{D.13}$$

and we obviously arrive at the conclusion that the flavour group is $\Delta(6n'^2)$.

For the remaining cases we use again the results of [44]. If 9 divides n, the flavour group is given by $\left(\mathbb{Z}_{n'} \times \mathbb{Z}_{n'/3}\right) \rtimes S_3$.

Finally, we have to consider $3 \mid n$ but $9 \nmid n$. In this case we put $\kappa = \omega^\alpha$ with $\alpha = pn/3$. Note that $\alpha = 1$ or $2 \pmod 3$. Now we proceed as before, but with the replacements $E \to \omega^{2\alpha} E$ in equation (D.3) and, due to equation (D.5),

$$\tilde{F} \to \omega^\alpha \tilde{F} = \text{diag}\left(\sigma'^2, \sigma'^*, \sigma'^*\right) \quad \text{with} \quad \sigma' = \omega^{-\alpha}\sigma. \tag{D.14}$$

Next we have a closer look at

$$\arg \sigma' = 2\pi \left(\frac{p}{n} - \frac{pn}{9}\right) = 2\pi \frac{p}{n} \left(1 - \left(\frac{n}{3}\right)^2\right). \tag{D.15}$$

Given that n is divisible by 3 but not by 9, we find that $1 - (n/3)^2$ is divisible by 3. So we end up with

$$\sigma' = \exp\left(2\pi i \, \frac{p\left[1 - (n/3)^2\right]/3}{n/3}\right). \tag{D.16}$$

This should be compared to $\sigma = e^{2\pi i p/n}$ defined before equation (D.8). In effect, we replace n by $n/3$ and we proceed as before, starting from equation (D.8). Eventually, we arrive at the group $\Delta\left(6(n'/3)^2\right)$, with n' being defined above.

E Two-flavour solutions

We are dealing here with

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \hat{\kappa}, \tag{E.1}$$

where θ is a rational angle, i.e. $e^{i\theta}$ is a root of unity. This case clearly involves only two-family mixing and we present it here only for completeness. Because it is physically irrelevant, we do not discuss all the details.

The phase factor κ_1 and the product $\kappa_2\kappa_3$ must be roots of unity. The only non-trivial part of T is the 23-sector, so we focus exclusively on this sector. There we have the two generators

$$\mathcal{T} \equiv \begin{pmatrix} \cos\theta\kappa_2 & \sin\theta\kappa_3 \\ -\sin\theta\kappa_2 & \cos\theta\kappa_3 \end{pmatrix}, \quad \mathcal{S} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{E.2}$$

where $\mathcal{S}$ is the 23-block of the generator S_2 of equation (1.9). We search for possible values of the pair $(\cos\theta, \kappa_2\kappa_3^*)$ which lead to a finite group. Let us denote the eigenvalues of $\mathcal{T}$ by a_1, a_2 and those of $\mathcal{TS}$ by b_1, b_2. Then we find

$$\mathrm{Tr}\mathcal{T} = a_1 + a_2 = \cos\theta\,(\kappa_2 + \kappa_3)\,, \quad \mathrm{Tr}(\mathcal{TS}) = b_1 + b_2 = \cos\theta\,(\kappa_2 - \kappa_3) \tag{E.3}$$

or

$$2\cos\theta\kappa_2 = \mathrm{Tr}\mathcal{T} + \mathrm{Tr}(\mathcal{TS}), \quad 2\cos\theta\kappa_3 = \mathrm{Tr}\mathcal{T} - \mathrm{Tr}(\mathcal{TS}). \tag{E.4}$$

Taking the square of the absolute values of these relations, we obtain consistency only if $(\mathrm{Tr}\mathcal{T})^*\,\mathrm{Tr}(\mathcal{TS})$ is purely imaginary. Finally, we can express $\cos^2\theta$ and $\kappa_2\kappa_3^*$ as

$$4\cos^2\theta = |\mathrm{Tr}\mathcal{T}|^2 + |\mathrm{Tr}(\mathcal{TS})|^2\,, \tag{E.5}$$

$$\kappa_2\kappa_3^* = \frac{|\mathrm{Tr}\mathcal{T}|^2 - |\mathrm{Tr}(\mathcal{TS})|^2 \pm 2i\,|\mathrm{Tr}\mathcal{T}|\,|\mathrm{Tr}(\mathcal{TS})|}{|\mathrm{Tr}\mathcal{T}|^2 + |\mathrm{Tr}(\mathcal{TS})|^2} \tag{E.6}$$

We can use equation (E.5) to constrain the possible values $|\mathrm{Tr}\mathcal{T}|^2$ and $|\mathrm{Tr}(\mathcal{TS})|^2$. Using $4\cos^2\theta = e^{-2i\theta} + e^{2i\theta} + 2$, we get the following vanishing sum of roots of unity:

$$0 = 2 - e^{2i\theta} - e^{-2i\theta} + a_1 a_2^* + a_1^* a_2 + b_1 b_2^* + b_1^* b_2. \tag{E.7}$$

This sum is exactly of the form of equation (3.10) of section 3, with its solutions given in equation (3.11). These solutions imply

$$\left(-e^{2i\theta},\, a_1 a_2^*,\, b_1 b_2^*\right) = (i, \omega, \omega)\,, \quad (\omega, \beta, \beta^2)\,, \quad \text{or} \quad (-1, \lambda, -\lambda)\,, \tag{E.8}$$

up to conjugations and permutations, where λ is some arbitrary root of unity. With this we obtain the seven cases

$$\begin{aligned} &\left(|\mathrm{Tr}\mathcal{T}|^2\,, |\mathrm{Tr}(\mathcal{TS})|^2\right) = \\ &(1,1)\,,\ (1,2)\,,\ \left(1, 2+\beta+\beta^4\right),\ \left(1, 2+\beta^2+\beta^3\right),\ \left(2+\beta+\beta^4, 2+\beta^2+\beta^3\right), \\ &(0, 2+\lambda+\lambda^*)\,,\ (2+\lambda+\lambda^*, 2-\lambda-\lambda^*) \end{aligned} \tag{E.9}$$

or the reversed order.

Note that $|\mathrm{Tr}\mathcal{T}|^2 \leftrightarrow |\mathrm{Tr}(\mathcal{TS})|^2$ preserves $\cos^2\theta$ — see equation (E.5) — and $\kappa_2\kappa_3^*$ is transformed to $-(\kappa_2\kappa_3^*)^*$ — see equation (E.6). While the conjugation of this phase is irrelevant, in general this is not true for the sign reversal. So, for each of the seven cases in equation (E.9), one has to take into account the two signs of $\kappa_2\kappa_3^*$.

A related concern would be the signs of $\cos\theta$ and $\sin\theta$, as equation (E.5) only tells us the value of $\cos^2\theta$. However, it is easy to see that such signs do not affect $|U|^2$.

In the following, we will list the solutions for each of the seven cases in equation (E.9) and the two sign variations of $\kappa_2\kappa_3^*$, without going into detail. For each of the seven cases we will display

1. $\cos^2\theta$ and $\kappa_2\kappa_3^*$,
2. $\hat{\kappa}$ and the eigenvalues of T for the positive sign,
3. $|U|^2$ for the positive sign,
4. the degenerate case of $|U|^2$ for the positive sign,
5. $\hat{\kappa}$ and the eigenvalues of T for the negative sign,
6. $|U|^2$ for the negative sign,
7. the degenerate case of $|U|^2$ for the negative sign.

First subcase. $\left(|\mathrm{Tr}\mathcal{T}|^2, |\mathrm{Tr}(\mathcal{T}\mathcal{S})|^2\right) = (1,1)$
This corresponds

$$\cos^2\theta = \frac{1}{2}, \quad \kappa_2\kappa_3^* = \pm i. \tag{E.10}$$

Since here $\kappa_2\kappa_3^*$ is purely imaginary, its sign variation is irrelevant.
Non-degenerate case:

$$\hat{\kappa} = \mathrm{diag}\,(\kappa_1, i\kappa_3, \kappa_3)\,, \quad \hat{\lambda}^{(0)} = \mathrm{diag}\,\left(\kappa_1, -\omega^2 e^{i\pi/4}\kappa_3, -\omega e^{i\pi/4}\kappa_3\right), \tag{E.11}$$

$$\mathcal{C}_{18}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6}\left(3+\sqrt{3}\right) & \frac{1}{6}\left(3-\sqrt{3}\right) \\ 0 & \frac{1}{6}\left(3-\sqrt{3}\right) & \frac{1}{6}\left(3+\sqrt{3}\right) \end{pmatrix}. \tag{E.12}$$

Degenerate case:

$$\hat{\lambda}^{(0)} = \kappa_1\,\mathrm{diag}\,(\omega, 1, 1)\,, \tag{E.13}$$

$$\mathcal{P}_3: \quad |U|^2 = \begin{pmatrix} 0 & \frac{1}{6}\left(3+\sqrt{3}\right) & \frac{1}{6}\left(3-\sqrt{3}\right) \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{E.14}$$

Second subcase. $\left(|\mathrm{Tr}\mathcal{T}|^2, |\mathrm{Tr}(\mathcal{T}\mathcal{S})|^2\right) = (1,2)$
This corresponds to

$$\cos^2\theta = \frac{3}{4}, \quad \kappa_2\kappa_3^* = \pm\frac{1}{3}\left(-1+2i\sqrt{2}\right) \equiv \pm\left(F_1\right)^2. \tag{E.15}$$

This means that we can write the phases as $\kappa_2 = F_1\kappa$, $\kappa_3 = F_1^*\kappa$ with a root of unity κ.
Positive sign, non-degenerate case:

$$\hat{\kappa} = \text{diag}\left(\kappa_1, F_1\kappa, F_1^*\kappa\right), \quad \hat{\lambda}^{(0)} = \text{diag}\left(\kappa_1, -\omega^2\kappa, -\omega\kappa\right), \tag{E.16}$$

$$\mathcal{C}_{19}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6}\left(3+\sqrt{6}\right) & \frac{1}{6}\left(3-\sqrt{6}\right) \\ 0 & \frac{1}{6}\left(3-\sqrt{6}\right) & \frac{1}{6}\left(3+\sqrt{6}\right) \end{pmatrix}. \tag{E.17}$$

Positive sign, degenerate case:

$$\hat{\lambda}^{(0)} = \kappa_1 \,\text{diag}\left(\omega, 1, 1\right), \tag{E.18}$$

$$\mathcal{P}_4: \quad |U|^2 = \begin{pmatrix} 0 & \frac{1}{6}\left(3+\sqrt{6}\right) & \frac{1}{6}\left(3-\sqrt{6}\right) \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{E.19}$$

Negative sign, non-degenerate case:

$$\hat{\kappa} = \text{diag}\left(\kappa_1, -F_1\kappa, F_1^*\kappa\right), \quad \hat{\lambda}^{(0)} = \text{diag}\left(\kappa_1, -ie^{-i\pi/4}\kappa, e^{-i\pi/4}\kappa\right) \tag{E.20}$$

$$\mathcal{C}_{20}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4}\left(2+\sqrt{2}\right) & \frac{1}{4}\left(2-\sqrt{2}\right) \\ 0 & \frac{1}{4}\left(2-\sqrt{2}\right) & \frac{1}{4}\left(2+\sqrt{2}\right) \end{pmatrix}. \tag{E.21}$$

Negative sign, degenerate case:

$$\hat{\lambda}^{(0)} = \kappa_1 \,\text{diag}\left(i, 1, 1\right), \tag{E.22}$$

$$\mathcal{P}_5: \quad |U|^2 = \begin{pmatrix} 0 & \frac{1}{4}\left(2-\sqrt{2}\right) & \frac{1}{4}\left(2+\sqrt{2}\right) \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{E.23}$$

Third subcase. $\left(|\text{Tr}\mathcal{T}|^2, |\text{Tr}(\mathcal{T}\mathcal{S})|^2\right) = (1, 2+\beta+\beta^4)$
This corresponds to

$$\cos^2\theta = \frac{1}{8}\left(5+\sqrt{5}\right), \quad \kappa_2\kappa_3^* = \pm\frac{1-2i}{\sqrt{5}} \equiv \pm\left(F_2\right)^2. \tag{E.24}$$

Positive sign, non-degenerate case:

$$\hat{\kappa} = \text{diag}\left(\kappa_1, F_2\kappa, F_2^*\kappa\right), \quad \hat{\lambda}^{(0)} = \text{diag}\left(\kappa_1, -\beta^3\kappa, -\beta^2\kappa\right), \tag{E.25}$$

$$\mathcal{C}_{21}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{20}\left(10-\sqrt{10\left(5+\sqrt{5}\right)}\right) & \frac{1}{20}\left(10+\sqrt{10\left(5+\sqrt{5}\right)}\right) \\ 0 & \frac{1}{20}\left(10+\sqrt{10\left(5+\sqrt{5}\right)}\right) & \frac{1}{20}\left(10-\sqrt{10\left(5+\sqrt{5}\right)}\right) \end{pmatrix}. \tag{E.26}$$

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Positive sign, degenerate case:

$$\hat{\lambda}^{(0)} = \kappa_1 \,\text{diag}\,(\beta, 1, 1)\,, \tag{E.27}$$

$$\mathcal{P}_6: \quad |U|^2 = \begin{pmatrix} 0 & \frac{1}{20}\left(10 - \sqrt{10\,(5+\sqrt{5})}\right) & \frac{1}{20}\left(10 + \sqrt{10\,(5+\sqrt{5})}\right) \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{E.28}$$

Negative sign, non-degenerate case:

$$\hat{\kappa} = \text{diag}\,(\kappa_1, -F_2\kappa, F_2^*\kappa)\,, \quad \hat{\lambda}^{(0)} = \text{diag}\,\left(\kappa_1, -i\omega^2\kappa, -i\omega\kappa\right), \tag{E.29}$$

$$\mathcal{C}_{22}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{12}\,(6+\sqrt{3}+\sqrt{15}) & \frac{1}{12}\,(6-\sqrt{3}-\sqrt{15}) \\ 0 & \frac{1}{12}\,(6-\sqrt{3}-\sqrt{15}) & \frac{1}{12}\,(6+\sqrt{3}+\sqrt{15}) \end{pmatrix}. \tag{E.30}$$

Negative sign, degenerate case:

$$\hat{\lambda}^{(0)} = \kappa_1 \,\text{diag}\,(\omega, 1, 1)\,, \tag{E.31}$$

$$\mathcal{P}_7: \quad |U|^2 = \begin{pmatrix} 0 & \frac{1}{12}\,(6+\sqrt{3}+\sqrt{15}) & \frac{1}{12}\,(6-\sqrt{3}-\sqrt{15}) \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{E.32}$$

Fourth subcase. $\left(|\text{Tr}\mathcal{T}|^2\,, |\text{Tr}(\mathcal{T}\mathcal{S})|^2\right) = (1, 2+\beta^2+\beta^3)$
This corresponds to

$$\cos^2\theta = \frac{1}{8}\left(5-\sqrt{5}\right), \quad \kappa_2\kappa_3^* = \pm\frac{1+2i}{\sqrt{5}} \equiv \pm\,(F_3)^2\,. \tag{E.33}$$

Positive sign, non-degenerate case:

$$\hat{\kappa} = \text{diag}\,(\kappa_1, F_3\kappa, F_3^*\kappa)\,, \quad \hat{\lambda}^{(0)} = \text{diag}\,\left(\kappa_1, -\omega^2\kappa, -\omega\kappa\right), \tag{E.34}$$

$$\mathcal{C}_{23}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{12}\,(6-\sqrt{3}+\sqrt{15}) & \frac{1}{12}\,(6+\sqrt{3}-\sqrt{15}) \\ 0 & \frac{1}{12}\,(6+\sqrt{3}-\sqrt{15}) & \frac{1}{12}\,(6-\sqrt{3}+\sqrt{15}) \end{pmatrix}. \tag{E.35}$$

Positive sign, degenerate case:

$$\hat{\lambda}^{(0)} = \kappa_1 \,\text{diag}\,(\omega, 1, 1)\,, \tag{E.36}$$

$$\mathcal{P}_8: \quad |U|^2 = \begin{pmatrix} 0 & \frac{1}{12}\,(6-\sqrt{3}+\sqrt{15}) & \frac{1}{12}\,(6+\sqrt{3}-\sqrt{15}) \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{E.37}$$

Negative sign, non-degenerate case:

$$\hat{\kappa} = \text{diag}\,(\kappa_1, -F_3\kappa, F_3^*\kappa)\,, \quad \hat{\lambda}^{(0)} = \text{diag}\,\left(\kappa_1, -i\beta^4\kappa, -i\beta\kappa\right), \tag{E.38}$$

$$\mathcal{C}_{24}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{20}\left(10+\sqrt{50-10\sqrt{5}}\right) & \frac{1}{20}\left(10-\sqrt{50-10\sqrt{5}}\right) \\ 0 & \frac{1}{20}\left(10-\sqrt{50-10\sqrt{5}}\right) & \frac{1}{20}\left(10+\sqrt{50-10\sqrt{5}}\right) \end{pmatrix}. \tag{E.39}$$

Negative sign, degenerate case:

$$\hat{\lambda}^{(0)} = \kappa_1 \,\mathrm{diag}\,\left(\beta^2, 1, 1\right), \tag{E.40}$$

$$\mathcal{P}_9: \quad |U|^2 = \begin{pmatrix} 0 & \frac{1}{20}\left(10-\sqrt{50-10\sqrt{5}}\right) & \frac{1}{20}\left(10+\sqrt{50-10\sqrt{5}}\right) \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{E.41}$$

Fifth subcase. $\left(|\mathrm{Tr}\mathcal{T}|^2\,, |\mathrm{Tr}(\mathcal{T}\mathcal{S})|^2\right) = (2+\beta+\beta^4, 2+\beta^2+\beta^3)$
This corresponds to

$$\cos^2\theta = \frac{3}{4}, \quad \kappa_2\kappa_3^* = \pm\frac{1}{3}\sqrt{1+4i\sqrt{5}} \equiv \pm\,(F_4)^2\,. \tag{E.42}$$

Positive sign, non-degenerate case:

$$\hat{\kappa} = \mathrm{diag}\,(\kappa_1, F_4\kappa, F_4^*\kappa)\,, \quad \hat{\lambda}^{(0)} = \mathrm{diag}\,\left(\kappa_1, -\beta^3\kappa, -\beta^2\kappa\right), \tag{E.43}$$

$$\mathcal{C}_{25}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{20}\left(10+\sqrt{50-10\sqrt{5}}\right) & \frac{1}{20}\left(10-\sqrt{50-10\sqrt{5}}\right) \\ 0 & \frac{1}{20}\left(10-\sqrt{50-10\sqrt{5}}\right) & \frac{1}{20}\left(10+\sqrt{50-10\sqrt{5}}\right) \end{pmatrix}. \tag{E.44}$$

Positive sign, degenerate case:

$$\hat{\lambda}^{(0)} = \kappa_1 \,\mathrm{diag}\,(\beta, 1, 1)\,, \tag{E.45}$$

$$\mathcal{P}_{10}: \quad |U|^2 = \begin{pmatrix} 0 & \frac{1}{20}\left(10+\sqrt{50-10\sqrt{5}}\right) & \frac{1}{20}\left(10-\sqrt{50-10\sqrt{5}}\right) \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{E.46}$$

Negative sign, non-degenerate case:

$$\hat{\kappa} = \mathrm{diag}\,(\kappa_1, -F_4\kappa, F_4^*\kappa)\,, \quad \hat{\lambda}^{(0)} = \mathrm{diag}\,\left(\kappa_1, -i\beta^4\kappa, -i\beta\kappa\right), \tag{E.47}$$

$$\mathcal{C}_{26}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{20}\left(10+\sqrt{50+10\sqrt{5}}\right) & \frac{1}{20}\left(10-\sqrt{50+10\sqrt{5}}\right) \\ 0 & \frac{1}{20}\left(10-\sqrt{50+10\sqrt{5}}\right) & \frac{1}{20}\left(10+\sqrt{50+10\sqrt{5}}\right) \end{pmatrix}. \tag{E.48}$$

Negative sign, degenerate case:

$$\hat{\lambda}^{(0)} = \kappa_1 \,\mathrm{diag}\,\left(\beta^2, 1, 1\right), \tag{E.49}$$

$$\mathcal{P}_{11}: \quad |U|^2 = \begin{pmatrix} 0 & \frac{1}{20}\left(10-\sqrt{50+10\sqrt{5}}\right) & \frac{1}{20}\left(10+\sqrt{50+10\sqrt{5}}\right) \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{E.50}$$

Sixth subcase. $\left(|\mathrm{Tr}\mathcal{T}|^2\,, |\mathrm{Tr}(\mathcal{T}\mathcal{S})|^2\right) = (0, 2+\lambda+\lambda^*)$
With $\lambda = e^{i\vartheta}$ unrestricted, except for being a root of unity, this corresponds to

$$\cos^2\theta = \cos^2\frac{\vartheta}{2}, \quad \kappa_2\kappa_3^* = \pm 1. \tag{E.51}$$

Positive sign, non-degenerate case:

$$\hat{\kappa} = \mathrm{diag}\,(\kappa_1, \kappa_3, \kappa_3)\,, \quad \hat{\lambda}^{(0)} = \mathrm{diag}\,\left(\kappa_1, \kappa_3 e^{i\theta}, \kappa_3 e^{-i\theta}\right), \tag{E.52}$$

$$\mathcal{C}_{27}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{E.53}$$

Positive sign, degenerate cases:

$$\hat{\lambda}^{(0)} = \kappa_1\,\mathrm{diag}\,\left(e^{2i\theta}, 1, 1\right), \tag{E.54}$$

$$\mathcal{P}_{12}: \quad |U|^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}, \tag{E.55}$$

$$\hat{\lambda}^{(0)} = \mathrm{diag}\,(\kappa_1, \kappa_3, \kappa_3)\,, \tag{E.56}$$

$$\mathcal{P}_{13}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{E.57}$$

Negative sign, non-degenerate case:

$$\hat{\kappa} = \mathrm{diag}\,(\kappa_1, -\kappa_3, \kappa_3)\,, \quad \hat{\lambda}^{(0)} = \mathrm{diag}\,(\kappa_1, \kappa_3, -\kappa_3)\,, \tag{E.58}$$

$$\mathcal{C}_{28}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\frac{\theta}{2} & \cos^2\frac{\theta}{2} \\ 0 & \cos^2\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}. \tag{E.59}$$

Negative sign, degenerate case:

$$\hat{\lambda}^{(0)} = \kappa_1\,\mathrm{diag}\,(-1, 1, 1)\,, \tag{E.60}$$

$$\mathcal{P}_{14}: \quad |U|^2 = \begin{pmatrix} 0 & \cos^2\frac{\theta}{2} & \sin^2\frac{\theta}{2} \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \tag{E.61}$$

At this point a note is in order. The only restriction on the angle θ in $\mathcal{C}_{28}$ is that it must be a rational multiple of π, i.e. $\exp i\theta$ must be a root of unity. It is therefore natural to ask whether or not the $|U|^2$ in $\mathcal{C}_{18}$–$\mathcal{C}_{27}$ and $\mathcal{C}_{29}$ are particular cases of $\mathcal{C}_{28}$. It turns out that the mixing matrices associated with $\mathcal{C}_{20}$, $\mathcal{C}_{27}$ and $\mathcal{C}_{29}$ are indeed particular cases of the one of $\mathcal{C}_{28}$, with $\theta = 3\pi/4$, $\pi/2$ and $\theta = \pi$, respectively. For the remaining cases, theorem 5 can be used to show that the associated angles θ are not rational.

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Seventh subcase. $\left(|\mathrm{Tr}\mathcal{T}|^2, |\mathrm{Tr}(\mathcal{T}\mathcal{S})|^2\right) = (2+\lambda+\lambda^*, 2+\lambda+\lambda^*)$
This corresponds to

$$\cos^2\theta = 1, \quad \kappa_2\kappa_3^* = \pm\lambda. \tag{E.62}$$

Since λ is a generic root of unity, the sign is irrelevant here and κ_1, κ_2, κ_3 are different roots of unity leading to

$$\mathcal{C}_{29}: \quad |U|^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{E.63}$$

The degenerate case has already been treated in $\mathcal{P}_{13}$.

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